Weil's Conjecture on Tamagawa Number

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Contents

1. Introduction

1.1. **Introduction to Weil's conjecture.** In number theory, the Hasse principle is the idea that one can find integer solutions to an equation by combining together solutions modulo prime powers. This process is handled by considering the equation over all the completions of the rational numbers: the real numbers $\mathbb{R}$ and the p-adic numbers $\mathbb{Q}_p$. The adeles $\mathbb{A}$ of $\mathbb{Q}$ is a ring that combines all these completions together, with the purpose that instead of trying to do analysis over each completion separately, one should put them on an equal footing by simply working over the adeles. For a first concrete indication of this phenomenon, the adeles lies inside the product $\mathbb{R} \times \prod_p \mathbb{Q}_p$ of all completions of $\mathbb{Q}$. Many statements in number theory, such as class field theory, have more enlightening adelic formulations than their classical accounts.

For a linear algebraic group G over $\mathbb{Q}$ (for example, $\mathrm{SL}_n, \mathrm{GL}_n$), one can study analysis on the adelic points $G(\mathbb{A})$ of G. In particular, we have a *canonical* Haar measure on $G(\mathbb{A})$, called the *Tamagawa measure* $\mu_{G,\mathbb{Q}}$.

As $\mathbb{Q}$ is a discrete subgroup of $\mathbb{A}$, $G(\mathbb{Q})$ is also a discrete subgroup of $G(\mathbb{A})$. Hence, the Tamagawa measure on $G(\mathbb{A})$ induces a $G(\mathbb{A})$-invariant measure on $G(\mathbb{Q}) \setminus G(\mathbb{A})$. When G is semisimple (for example, SL_n or SO_n), the volume

$$\tau_{\mathbb{Q}}(G) := \int_{G(\mathbb{Q})\setminus G(\mathbb{A})} \mu_{G,\mathbb{Q}}.$$

of $G(\mathbb{Q}) \setminus G(\mathbb{A})$ with respect to this measure, called the *Tamagawa number*, is finite, and contains interesting arithmetic information. For example, $\tau_{\mathbb{Q}}(\mathrm{SL}_2) = 1$ is equivalent to the Euler product formula for the value of the Riemann zeta function ζ at 2. Tamagawa [Tam66] was the first one to define $\tau_k(G)$ when G is the special orthogonal group of a quadratic form and k is any number field (for example, $k = \mathbb{Q}$). He showed that $\tau_k(\mathrm{SO}_n) = 2$ and indicated that this is entirely equivalent to Siegel's famous mass formula in the theory of quadratic forms (we will try to motivate this idea of Tamagawa in a later section). Weil [Wei95] pursued Tamagawa's idea for more general groups and conjectured

Theorem 1 (Weil's conjecture on Tamagawa numbers)**.** *Let G be a simply connected semisimple linear algebraic group over a number field or a function field k, then the Tamagawa number $\tau_k(G)$ of G over k is* 1*.*

1.2. History of Weil's conjecture.

1.2.1. *Proofs.* In the number field case, Weil, Mars and Demazure (see [Wei82, p. 116] for the precise references) have settled the conjecture for all classical groups and some exceptional groups. Langlands [Lan66] proved the conjecture for split groups G over $\mathbb{Q}$. Lai [Lai76] generalised Langlands' proof to the quasi-split case over any number field. Kottwitz [Kot88] completely proved the conjecture by reducing to the quasi-split case via the Arthur-Selberg trace formula.

In the function field case, Harder [Har74] proved the conjecture for split groups following Langlands' method. Until much later, Gaitsgory and Lurie [GL19] proved the conjecture for the function field case using a completely different method.

1.2.2. *Beyond Weil's conjecture.* The Tamagawa measure and the Tamagawa number can be defined more generally for any connected reductive or unipotent linear algebraic group G over a global field k (see [Ono66]). Ono [Ono66] derived the Tamagawa number for tori and showed how the Tamagawa numbers behave under isogenies. In particular, consider a connected semisimple algebraic group G over a number field and let $\tilde{G}$ be its universal covering. Ono obtained a formula for $\tau(G)/\tau(\tilde{G})$, called the relative Tamagawa number. This was then generalised to any connected reductive groups over a number field by Sansuc [San81] and over a function field by [BD09]. In particular, Sansuc showed that for any connected reductive group G over a number field and $\tilde{G}'$ be the universal covering of its derived subgroup, we find

$$\frac{\tau(G)}{\tau(\tilde{G}')} = \frac{|\operatorname{Pic} G|}{|Ш(G)|}.$$

Here $\operatorname{Pic}(G)$ is the Picard group of G and $Ш(G)$ is the Tate-Shafarevich set of G. Assuming the Weil's conjecture, this implies

Theorem 2. *For a connected reductive algebraic group G over a global field k, there holds*

$$\tau_k(G) = \frac{|\operatorname{Pic}(G)|}{|Ш(G)|}.$$

When G is unipotent, the Tamagawa number of G over a number field is 1. Over function fields, the Tamagawa number was computed in [Oes84].

In the other direction, Birch and Swinnerton-Dyer attempted to find analogues of Tamagawa measure and Tamagawa number in the case of abelian varieties. This has led them to conjecture about the integer values of certain L-functions of elliptic curves. In particular, Bloch [Blo80] showed theorem

Theorem 3. *Let A be an abelian variety over a number field. There exists an extension X of A by a split torus such that one can define the Tamagawa measure and Tamagawa number for X. Furthermore, the Birch and Swinnerton-Dyer conjecture for A is equivalent to the statement*

$$\tau_k(X) = \frac{|\operatorname{Pic}(X)_{tors}|}{|Ш(X)|},$$

where $\operatorname{Pic}(X)_{tors}$ *is the torsion subgroup of* $\operatorname{Pic}(X)$.

1.3. **Motivation.** We will motivate the Tamagawa numbers via the arithmetic theory of quadratic forms, following [GL19].

A *quadratic space* (V, q) over $\mathbb{Q}$ is a finitely generated free $\mathbb{Q}$-module V equipped with a *quadratic form*, i.e. a map $q : V \to \mathbb{Q}$ satisfying the following conditions:

(1) The map $V \times V \to \mathbb{Q}$ given by $(v, w) \mapsto q(v + w) - q(v) - q(w)$ is $\mathbb{Q}$-bilinear.
(2) For every $\lambda \in \mathbb{Q}$ and every $v \in V$, we have $q(\lambda v) = \lambda^2 q(v)$.

A morphism between two quadratic spaces (V, q) and (V', q') is a linear map $f : V \to V'$ such that $q' \circ f = q$. The automorphism group of a quadratic space (V, q) over $\mathbb{Q}$ is denoted as $O_q(\mathbb{Q})$, the *orthogonal group* of (V, q).

If we fix a choice of basis $\{e_1, \ldots, e_n\}$ for V, a quadratic form q on V then corresponds to a matrix B_q defined by $(B_q)_{ij} = \frac{1}{2}(q(e_i + e_j) - q(e_i) - q(e_j))$. One can show that two quadratic forms p, q on V are isomorphic if and only if $B_p = T^t B_q T$ for some invertible matrix $T \in \operatorname{GL}_n(\mathbb{Q})$.

One could then ask the question of classifying quadratic spaces over $\mathbb{Q}$ up to isomorphism; or equivalently, classifying $n \times n$ matrices over $\mathbb{Q}$ up to the equivalence relation $A \sim B \iff A = T^t BT$ for some $T \in \operatorname{GL}_n(\mathbb{Q})$. To achieve this, one first base changes the quadratic space (V, q) over $\mathbb{Q}$ to create a quadratic space $(V \otimes_{\mathbb{Q}} \mathbb{Q}_v, q_{\mathbb{Q}_v})$ over $\mathbb{Q}_v$ for each completion $\mathbb{Q}_v$ of $\mathbb{Q}$ (we think of $\mathbb{R}$ as $\mathbb{Q}_\infty$). The Hasse principle for quadratic forms then states that two quadratic forms are equivalent over $\mathbb{Q}$ if and only if they are equivalent over $\mathbb{Q}_v$'s. Over $\mathbb{Q}_v$'s, the classification of quadratic spaces is easier to describe, giving us the classification over $\mathbb{Q}$ (see [Ser73, Chapter IV]).

If we now restrict our attention to quadratic spaces over $\mathbb{Z}$ then a similar statement to the Hasse principle fails; i.e. even if s and q are two quadratic forms over $\mathbb{Z}$ such that they equivalent under extension of scalars to $\mathbb{Z}_p$ and $\mathbb{R}$ (this is to say that s and q *have the same genus*), it does not necessarily follow that they are equivalent over $\mathbb{Z}$.

However, it is 'almost true' in the following sense: for a fixed positive-definite quadratic form q over $\mathbb{Z}$ [1], there are only finitely many quadratic spaces of the same genus to q (up to isomorphism). In fact, one obtains a bijection (or an equivalence of groupoids)

$$\{\text{genus of } q\} \longleftrightarrow O_q(\mathbb{Q}) \setminus O_q(\mathbb{A})/O_q(\widehat{\mathbb{Z}} \times \mathbb{R}).$$

Furthermore, we can also define the following weight count of quadratic forms in the genus of g, (called the *mass* of the genus of q)

$$m(q) = \sum_{q'} \frac{1}{|O_{q'}(\mathbb{Z})|}$$

where the sum is over all quadratic forms of the same genus as q up to isomorphism. One can show that

$$m(q) = \sum_{q'} \frac{1}{|O_{q'}(\mathbb{Z})|} = 2^{\ell-1} \frac{\mu_{\text{Tam}}(SO_q(\mathbb{Q}) \setminus SO_q(\mathbb{A}))}{\mu_{\text{Tam}}(SO_q(\widehat{\mathbb{Z}} \times \mathbb{R}))}.$$

Here μ_{Tam} is the Tamagawa measure on $SO_q(\mathbb{A})$, i.e. the group of automorphisms in $O_q(\mathbb{A})$ having determinant 1, ℓ is the number of primes p for which $SO_q(\mathbb{Z}_p) = O_q(\mathbb{Z}_p)$, and $\widehat{Z} := \prod_p \mathbb{Z}_p$. The numerator is the Tamagawa number for SO_q, and as one can also compute the denominator, this gives an explicit mass formula, called the Smith-Minkowski-Siegel mass formula.

1.4. **Outline for the thesis.** The goal of our thesis is to study the Weil's conjecture. In the next five sections, we will present the prerequisite back ground material. We will then define the Tamagawa measure in § 8 and compute the Tamagawa numbers for various groups in the next four sections. In the last section, we give an equivalent formulation for the Tamagawa number in the function field case. A more detailed outline for the thesis is given below.

In § 2, we discuss valuation theory, i.e. how to equip a field k with an absolute value and take completions of k with respect to this absolute value. Our main example is $k = \mathbb{Q}$ with its completions $\mathbb{Q}_p$ and $\mathbb{R}$, where p is a prime.

In § 3, first we review the theory of measures and integrations. We then focus on discussing Haar measures on locally compact topological groups and establish some results that will be used to carry out computations with Haar measures in later sections.

In § 4, we define the notion of a k-analytic manifold for any complete valued field k. When k is a local field (e.g. $\mathbb{R}$ or $\mathbb{Q}_p$), we show that there is a theory of integration on such manifolds, resembling the corresponding classical theory for smooth manifolds.

In § 5, we define the ring adeles $\mathbb{A}$ of $\mathbb{Q}$ and study its topology. We show that $\mathbb{Q}$ is a discrete subgroup of $\mathbb{A}$ and that $\mathbb{Q} \setminus \mathbb{A}$ is compact. We also describe a functorial way to give a topology on $G(\mathbb{A})$ for any linear algebraic group G over $\mathbb{Q}$.

In § 6, we discuss Fourier analysis on locally compact abelian groups. In particular, we describe the Pontryagin duals for $\mathbb{R}, \mathbb{Q}_p$ and $\mathbb{A}$ together with their quotients $\mathbb{Z} \setminus \mathbb{R}, \mathbb{Z}_p \setminus \mathbb{Q}_p$ and $\mathbb{Q} \setminus \mathbb{A}$, respectively. We then prove the Poisson summation formula with the focus on these groups.

In § 7, we discuss the structure theory of SL_2. We determine its Lie algebra and its non-vanishing left-invariant global top forms, and then show how to relate these two notions.

In § 8, we define the Tamagawa measure and the Tamagawa number for any connected semisimple group over any global field.

In § 9, we compute the Tamagawa number of SL_2 over $\mathbb{Q}$ by constructing a fundamental domain for the quotient $\mathrm{SL}_2(\mathbb{Q})\setminus \mathrm{SL}_2(\mathbb{A})$.

In § 10, § 11 and § 12, we will exhibit computations of the Tamagawa numbers of classical groups over $\mathbb{Q}$ using Poisson summation formula.

[1] q being positive-definite means $q_{\mathbb{R}}$ is positive-definite, i.e. $q_{\mathbb{R}}(v) > 0$ for every nonzero vector v

In § 13, we showed that the Tamagawa number over function field can be interpreted as certain weight count on the moduli space of G-bundles.

2. Absolute values, completions of $\mathbb{Q}$

In this section, following [Mil, Sut19, Neu99], we will discuss the completion of a field that is equipped with an absolute value. We focus on describing the completions $\mathbb{Q}_p$'s and $\mathbb{R}$ of $\mathbb{Q}$.

2.1. **Absolute values.**

Definition 4. *An* absolute value *of a field k is a map $|\cdot| : k \to \mathbb{R}_{\geq 0}$ such that for all $x, y \in k$:*

(1) $|x| = 0$ *iff* $x = 0$,
(2) $|xy| = |x||y|$,
(3) $|x + y| \leq |x| + |y|$.

The field k is then called a valued field. *If the stronger condition*

(4) $|x + y| \leq \max(|x|, |y|)$.

also holds, then the absolute value is nonarchimedean, *otherwise it is* archimedean.

The condition $|x + y| \leq \max(|x|, |y|)$ for all $x, y \in k$ is equivalent to $|\ |$ being bounded on $\{n1 : n \in \mathbb{Z}\}$. In particular, this implies that if k is of positive characteristic then every absolute value on k is nonarchimedean.

For valued field k with nonarchimedean absolute value $|\ |$, the set $\mathcal{O}_k := \{x \in k : |x| \leq 1\}$ is a subring of k with group of units $U := \{x \in k : |x| = 1\}$ and unique maximal ideal $\mathfrak{m} := \{x \in k : |x| < 1\}$.

Example 5. The map $|\ | : k \to \mathbb{R}_{\geq 0}$ defined by $|x| = 1$ if $x \neq 0$ and $|0| = 0$, is the *trivial absolute value* on k. It is nonarchimedean.

When k is equipped with an absolute value then k has a metric space topology. Two absolute values $|\ |_1$ and $|\ |_2$ on k are called equivalent if they define the same topology on k. This is the same as saying that there exists real number $s > 0$ such that $|x|_1^s = |x|_2$ for all $x \in k$. We call an equivalence class of absolute values on k a *place* of k.

For $\mathbb{Q}$, we have the usual absolute value $|\ |_\infty$, being an archimedean absolute value. For each prime p, we can define an archimedean absolute value $|\ |_p$ on $\mathbb{Q}$ as follows.

Example 6. Let p be a prime number. Every element in $\mathbb{Q}^\times$ can be written as $\pm \prod_q q^{e_q}$ where the product ranges over the primes of $\mathbb{Z}$ and the exponents $e_q \in \mathbb{Z}$ are uniquely determined. We have a map (called the p-adic valuation) $v_p : \mathbb{Q}^\times \to \mathbb{Z}$, defined by

$$v_p\left(\pm \prod_q q^{e_q}\right) = e_p,$$

and $v_p(0) := \infty$. The *p-adic absolute value* on $\mathbb{Q}$ is defined by $|x_p| := p^{-v_p(x)}$, where $|0|_p = p^{-\infty}$ is understood to be 0.

Theorem 7 (Ostrowski)**.** *Every nontrivial absolute value on $\mathbb{Q}$ is equivalent to either $|\ |_\infty$ or $|\ |_p$ for some prime p.*

Sketch. For any $m, n \in \mathbb{Z}$, we can write $m = a_0 + a_1 n + \cdots + a_r n^r$ where $a_i \in \mathbb{Z}, 0 \leq a_i < n, n^r \leq m$. Letting $N := \max\{1, |n|\}$, we obtain a bound $|m| \leq N^{\log m/\log n}$.

If for all $n > 1$, we have $|n| > 1$, then $N = |n|$. From the previous inequality, we find $|m|^{1/\log m}$ is constant for all $m \in \mathbb{Z}_{>1}$. It follows $|n| = |n|_\infty^{\log c}$ for all integer $n > 1$, hence $|\ |$ is equivalent to $|\ |_\infty$.

If there is $n \in \mathbb{Z}$ such that $n > 1$ but $|n| \leq 1$, then $N = 1$ and hence $|m| \leq 1$ for all $m \in \mathbb{Z}$, meaning the absolute value is nonarchimedean. Let $\mathcal{O}$ be the corresponding local ring and $\mathfrak{m}$ be its maximal ideal. We find $\mathbb{Z} \subset \mathcal{O}$ and $\mathfrak{p} \cap \mathbb{Z}$ is a nonzero prime ideal, hence this ideal is (p) for some prime p. This implies $|m| = 1$ if m is not divisible by p, hence $|ap^r| = |p|^r$ if n is rational number whose numerator and denominator are not divisible by p. If $a \in \mathbb{R}$ such that $|p| = (1/p)^a$ then $|x| = |x|_p^a$ for all $x \in \mathbb{Q}$. □

For a number field k, i.e. a finite extension of $\mathbb{Q}$, we can describe the *places* of k, i.e. equivalence classes of absolute values on k, as follows.

Theorem 8. *There exists exactly one place of k*

(1) for each prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$,
(2) for each real embedding of k (i.e. an injective field homomorphism $k \hookrightarrow \mathbb{R}$),
(3) for each conjugate pair of complex embeddings.

For a place v of k coming from an archimedean absolute value, we write $v \mid \infty$.

Example 9. When $k = \mathbb{Q}[x]/(x^2+1)$, we have one conjugate pair of complex embeddings $k \hookrightarrow \mathbb{C}$ sending $x \mapsto \pm i$. This corresponds to the completion $\mathbb{C}$ of k. On the other hand, the ring of integers $\mathcal{O} = \mathbb{Z}[x]/(x^2+1)$ of k has prime ideals

(1) $(1+i) = (1-i)$,
(2) $(a+ib)$ where $a^2+b^2 = p$ is a prime with $p \equiv 1 \pmod 4$,
(3) (p) where $p \in \mathbb{Z}$ is a prime such that $p \equiv 3 \pmod 4$.

The absolute value of k corresponding to each prime ideal is defined analogously as in the case of p-adic absolute value for $\mathbb{Q}$.

2.1.1. *Nonarchimedean absolute values from discrete valuations.* The class of nonarchimedean valued fields that is of interest for us comes from discrete valuations.

Definition 10. *A* valuation *on a field k is a group homomorphism $k^\times \to \mathbb{R}$ such that for all $x, y \in k^\times$*

$$v(x+y) \geq \min\{v(x), v(y)\}.$$

We may extend v to a map $k \to \mathbb{R} \cup \{\infty\}$ by defining $v(0) := \infty$. For any $0 < c < 1$, defining $|x|_v := c^{v(x)}$ yields the same nonarchimedean absolute value up to equivalence. We say v is a (normalised) discrete valuation *if $v(k^\times) = \mathbb{Z}$. We call $A := \{x \in k : v(x) \geq 0\}$ the* valuation ring *of k. A* discrete valuation ring *is an integral domain that is the valuation ring of its fraction field with respect to a discrete valuation.*

Example 11. For $k = \mathbb{Q}$, the p-adic absolute value comes from the discrete valuation v_p as in Example 6.

For a discrete valuation ring A, there holds $v(A) = \mathbb{Z}_{\geq 0}$, so there exists elements $\pi \in A$ such that $v(\pi) = 1$, which we call them *uniformisers* of A. If we fix a uniformiser π then every element $x \in k^\times$ can be written uniquely as $x = u\pi^n$, where $n = v(x)$ and $u = x/\pi^{v(x)} \in A^\times$. Every nonzero ideal of A is equal to $(\pi^n) = \{a \in A : v(a) \geq n\}$ for some integer $n \geq 0$. Hence, A has a unique maximal ideal $\mathfrak{m} = (\pi) = \{a \in A : v(a) \geq 0\}$.

A discrete valuation ring enjoys many properties which gives it many equivalent definitions. At the moment, we will direct the reader to [Sut19, Lecture 1], [Ser79], [Mil] for further discussions about this.

Example 12. The p-adic valuation v_p of $\mathbb{Q}$ as in Example 6 has valuation ring $\mathbb{Z}_{(p)}$, which is the localisation of $\mathbb{Z}$ at the multiplicative set $\mathbb{Z} \setminus (p)$. Concretely, it is a subring of $\mathbb{Q}$, with elements of the form $\frac{a}{b} \in \mathbb{Q}$ where $p \nmid b$. The residue field is $\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)} \cong \mathbb{Z}/p\mathbb{Z} \cong \mathbb{F}_p$.

Example 13. For any field k, the valuation $v : k((t)) \to \mathbb{Z} \cup \{\infty\}$ on the field of Laurent series over k defined by

$$v\left(\sum_{n \geq n_0} a_n t^n\right) := n_0,$$

where $a_{n_0} \neq 0$, has valuation ring $k[[t]]$, the ring of power series with coefficients in k. For $f \in k((t))^\times$, $v(f) \in \mathbb{Z}$ is the order of vanishing of f at 0.

2.2. **Completions of global fields.**

Definition 14. *Let k be a field with nontrivial absolute value. A sequence (a_n) of elements in k is called a* Cauchy sequence *if for every $\varepsilon > 0$, there is $N > 0$ such that $|a_n - a_m| < \varepsilon$ for all $n, m > N$. The field k is said to be* complete *if every Cauchy sequence has a limit in k.*

Theorem 15. *Let k be a field with absolute value $| \ |$. There exists a complete valued field $(\hat{k}, | \ |)$ and a homomorphism $k \to \hat{k}$ of topological fields, preserving the absolute value, that is universal in the following sense: every homomorphism $k \to l$ from k to a complete valued field $(l, | \ |)$ preserving the absolute value extends uniquely to a homomorphism $\hat{k} \to l$.*

Sketch. Construct $\hat{k}$ to be the set of equivalence classes of Cauchy sequences in $\hat{k}$, in the sense that two Cauchy sequences (a_n) and (b_n) are equivalent when $\lim_{n\to\infty} |a_n - b_n| = 0$. One can then define addition and multiplication in the obvious way and show that $\hat{k}$ is a field. An element $a \in k$ has image $(a, a, \dots)$ inside $\widehat{k}$. □

We are interested in completed valued fields that come from taking completions of a *global field*, i.e. a finite extension of $\mathbb{Q}$ or of $\mathbb{F}_q((t))$. The resulting completed fields are called local fields, which have the following equivalent but simple description.

Definition 16. *A* local field *is a valued field k with nontrivial absolute value such that k is locally compact.*

Note that if k is locally compact then k is complete [2]. All archimedean local fields are isomorphic to either $\mathbb{R}$ or $\mathbb{C}$.

2.2.1. *Completions from discrete valuations.* This section is about complete valued fields with discrete valuation, which, in particular, is where all nonarchimedean local fields come from.

Let $| \ |$ be a nonarchimedean absolute value on k obtained via a discrete valuation v. Let $A, \mathfrak{m}, \pi$ be the corresponding valuation ring, maximal ideal and uniformiser of k.

Proposition 17. *(a) If $\widehat{k}$ is the completion of k with respect to $| \ |$ then $| \ |$ is also a discrete absolute value on $\widehat{k}$. Its maximal ideal $\widehat{\mathfrak{m}}$ is generated by π. The residue field of $\widehat{k}$ is $A/\mathfrak{m} \cong \widehat{A}/\widehat{\mathfrak{m}}$.*

(b) If $S \subset A$ is a set of representatives of $A/\mathfrak{m}$ then every element in $\widehat{k}$ has a unique representative of the form

$$a_{-n}\pi^{-n} + \cdots + a_0 + a_0\pi + \cdots + a_m\pi^m + \dots, \qquad a_i \in S.$$

(c) Furthermore, we have an isomorphism of topological rings

$$\widehat{A} \cong \varprojlim_{n\to\infty} \frac{A}{\pi^n A}.$$

[2]Let $(x_n)_{n=1}^\infty$ be a sequence in k that converges to $x \in \widehat{k}$. Let $U \subset k$ be a compact neighborhood of x_1 then $x_n x_1^{-1} U$ is a compact neighborhood of x_n. We should be able to find $x \in \bigcup_n x_n x_1^{-1} U \subset k$.

Sketch. (a) Let $a \in \widehat{k}^\times$ then a corresponds to a sequence (a_n) in k converging to a. Then $|a_n| \to |a|$, so $|a|$ is a limit point of $|k^\times|$. But $|k^\times|$ is discrete of $\mathbb{R}$, hence closed, hence $|a| \in |k^\times|$. Thus $|\ |$ is a discrete absolute value on $\widehat{k}$ and also use v to denote the valuation on $\widehat{k}$ extending the one on k. It follows that $\widehat{\mathfrak{m}}$ is generated by π.

(b) Let $\alpha \in \widehat{k}$ then $\alpha = \pi^n \alpha_0$ for α_0 unit in $\widehat{A}$. There exists $a_0 \in S$ such that $\alpha_0 - a_0 \in \widehat{\mathfrak{m}}$. Then $\frac{\alpha_0 - a_0}{\pi} \in \widehat{A}$ so there exists $a_1 \in S$ such that $\frac{\alpha_0 - a_0}{\pi} - a_1 \in \widehat{\mathfrak{m}}$. If we keep going then we can write $\alpha_0 = a_0 + a_1\pi + \dots$ and $\alpha = \pi^n \alpha_0$.

We refer to [Sut19, Lecture 8] for the proof of part (c). ☐

Example 18. Let $k = \mathbb{Q}$, v_p be the p-adic valuation of $\mathbb{Q}$ and $|x|_p := p^{-v_p(x)}$ be the corresponding p-adic absolute value. The completion of $\mathbb{Q}$ with respect to $|\ |_p$ is the field $\mathbb{Q}_p$ of p-adic numbers. For $x = a_m p^m + a_{m+1}p^{m+1} \cdots \in \mathbb{Q}_p$ where $m \in \mathbb{Z}, a_i \in \mathbb{F}_p, a_m \neq 0$ then $|x|_p = p^{-m}$. From Example 12, v_p over $\mathbb{Q}$ has valuation ring $\mathbb{Z}_{(p)}$, and we have $\widehat{\mathbb{Z}_{(p)}} = \mathbb{Z}_p$, the p-adic integers. The basis of open sets of $0 \in \mathbb{Q}_p$ are $p^k\mathbb{Z}_p$ where $k \in \mathbb{Z}$.

Example 19. Let $k = k(t)$, let v_t be the t-adic valuation on $k(t)$, and let $|x|_t := q^{-v_t(x)}$ (for $q > 1$ any fixed real number) be the corresponding absolute value with $\pi = t$ being the uniformiser. The completion of $k(t)$ with respect to $|\ |_t$ is isomorphic to field $k((t))$ of Laurent series over k. The valuation ring of $k(t)$ with respect to v_t is $k[t]_{(t)}$, ring of rational functions whose denominators have nonzero constant term. With $\pi = t$ as our uniformiser, we find $\widehat{k[t]_{(t)}} = k[[t]]$, the power series over k.

Proposition 20. *k is locally compact if and only if it is complete and has finite residue field $A/\mathfrak{m}$.*

Proof. If k is locally compact then k is complete. As $\{\pi^n A\}_{n\in\mathbb{Z}}$ is a fundamental system of closed neighborhoods of 0, at least one of them is compact. Multiplying by π^{-n}, which is a homeomorphism, shows that A is compact. Let S be set of representatives for $A/\mathfrak{m}$, then the compact subset A is a disjoint union of open sets $s + \mathfrak{m}$ for $s \in S$, implying S is finite.

Conversely, if $A/\mathfrak{m}$ is finite then $A/\pi^n A$ is finite, hence from previous proposition, $\widehat{A}$ is a projective limit of finite rings, hence is compact. If k is complete then $A = \widehat{A}$ is compact, meaning k is locally compact. ☐

Example 21. 1) The completion $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to p-adic valuation v_p is locally compact, hence a nonarchimedean local field.

2) $\mathbb{F}_q((t))$ is locally compact as it is the completion of $\mathbb{F}_q(t)$ with respect to t-adic valuation and residue field $\mathbb{F}_q$.

3. Measures and integration

In this section we review the theory of measure spaces and integration on locally compact spaces, in particular Haar measure on locally compact groups. We refer to [Fol16, VR99, Kna02, BSU96] for the proofs of the results in this section.

Convention 22. From now on, all locally compact spaces are assumed to be Hausdorff.

3.1. **Measure.**

Let X be a set, and let $\mathcal{M}$ be a collection of subsets of X.

Definition 23. $\mathcal{M}$ *is a* σ-algebra *if* $\mathcal{M}$ *is closed under taking complements in* X *and countable unions. Elements of* $\mathcal{M}$ *are called* measurable sets*.*

Example 24. Let X be a topological space. The collection of *Borel sets* is the σ-algebra $\mathcal{B}(X)$ generated by open subsets of X.

Definition 25. *A function* $f : X \to Y$ *is called* measurable *if the preimage of any measurable subset in* Y *is measurable in* X*.*

Remark 26. Let f be a complex-valued function on a σ-algebra, where the measurable sets in $\mathbb{C}$ are the Borel sets of $\mathbb{C}$. For f to be measurable, it suffices to check $f^{-1}(S)$ is measurable for open disks in $\mathbb{C}$. When f is real-valued, f is measurable iff $f^{-1}(S)$ is measurable for any $S = (a, \infty) \subset \mathbb{R}$ where $a \in \mathbb{R}$.

Definition 27. *A* measure *on* $(X, \mathcal{M})$ *is a function* $\mu : \mathcal{M} \to [0, \infty]$ *such that* $\mu\left(\bigcup A_i\right) = \sum \mu(A_i)$ *for any countable (or finite) collection of disjoint measurable sets* A_i*. In the special case where* $\mathcal{M} = \mathcal{B}$*, a measure is called a* Borel measure*.*

A set $N \subset X$ is called a *null set* if N is contained in a measure-0 set. It is convenient to enlarge $\mathcal{M}$ so that all null sets are measurable. We call $f : X \to \mathbb{C}$ a *null function* if $\{x \in X : f(x) \neq 0\}$ is a null set.

Given a measure (X, μ) and a measurable map $f : X \to Y$ then the *pushforward* of μ is a measure on Y where $(f_*\mu)(B) := \mu(f^{-1}(B))$ for any measurable subset B of Y. We are not aware of any reference discussing pullback of measures in general. However, if X, Y are smooth manifolds and f is submersive, pullback of measures can be defined via fiber integrations.

3.2. **Integration.**

We fix the notation $(X, \mathcal{M}, \mu)$ where X is a set with σ-algebra $\mathcal{M}$ and measure μ. We will briefly define integration with respect to this space. We refer to [BSU96] for a more detailed discussion of this construction.

Given $S \in \mathcal{M}$ with $\mu(S) < \infty$, let $1_S : X \to \{0, 1\}$ be the indicator function on S, i.e. it has value 1 on S and 0 outside of S, and define $\int_X 1_S d\mu := \mu(S)$. A *simple* function f is a function of the form $f = \sum_{i=1}^n a_i 1_{S_i}$ where $a_i \in \mathbb{R}$ and S_i's are pairwise disjoint sets in $\mathcal{M}$ of finite measure. For such a simple function $f = \sum a_i 1_{S_i}$, define $\int_X f d\mu := \sum_i a_i \mu(S_i)$. For any real-valued nonnegative measurable function f on X, we define

$$\int_X f(x) d\mu(x) := \sup_\phi \int_X \phi(x) d\mu(x),$$

where ϕ ranges over all simple functions on X with $0 \leq \phi \leq f$. We say that a measurable function $f : X \to \mathbb{C}$ is *integrable* if $\int_X |f(x)| dx < \infty$. If f is integrable, we can write $f = (u^+ - u^-) + i(v^+ - v^-)$ where $u^+(x) = \max\{\mathrm{Re}(f(x)), 0\}$, $u^-(x) = -\min\{\mathrm{Re} f(x), 0\}$ and similarly for v^+, v^-. We then define

$$\int_X f(x) d\mu(x) := \int_X u^+ d\mu - \int_X u^- d\mu + i \int_X v^+ d\mu - i \int_X v^- d\mu.$$

We define $L^1(X, \mu)$ to be the Banach space of measurable functions $f : X \to \mathbb{C}$ that have finite L^1-norm $\|f\|_1 := \int_X |f| d\mu$.

Proposition 28 (Change of variables formula)**.** *Given a measure (X, μ) and a measurable map $f : X \to Y$. Then, for a measurable function $g : Y \to \mathbb{C}$, $g \circ f$ is measurable and*

$$\int_Y g d(f_*\mu) = \int_X g \circ f d\mu.$$

3.3. **Measures and integrals on locally compact Hausdorff space.** Let X be a locally compact topological space.

Definition 29. *A function $f : X \to \mathbb{C}$ has* compact support *if the closure of $\{x \in X : f(x) \neq 0\}$ is compact. Define $C_c(X)$ to be space of continuous functions $f : X \to \mathbb{C}$ of compact support.*

Definition 30. *An* outer Radon measure *on X is a Borel measure $\mu : \mathcal{B} \to [0, \infty]$ that is*

- *locally finite: every $x \in X$ has an open neighborhood U such that $\mu(U) < \infty$*
- *outer regular: every $S \in \mathcal{B}$ satisfies $\mu(S) = \inf \mu(U)$ over all open $U \supset S$,*
- *inner regular on open sets: every open $U \subset X$ satisfies $\mu(U) = \sup \mu(K)$ over all compact $K \subset U$.*

A Radon integral *on X is a $\mathbb{C}$-linear map $I : C_c(X) \to \mathbb{C}$ such that $I(f) \geq 0$ whenever $f \geq 0$.*

For a Radon measure space (X, μ), $C_c(X)$ is a subspace of $L^1(X, \mu)$.

Theorem 31 (Riesz representation theorem)**.** *Given an outer Radon measure μ, we define a linear functional*

$$I_\mu : C_c(X) \to \mathbb{C}$$
$$f \mapsto \int_X f d\mu.$$

When X is locally compact Hausdorff, there is a bijection between outer Radon measures on X and Radon integrals on X, where one direction is given by $\mu \mapsto I_\mu$. The other direction is by sending $I : C_c(X) \to \mathbb{C}$ to the measure μ on X defined by $\mu(S) = I(1_S)$.

Example 32. Let $X = \mathbb{R}^n$, the map sending $f \in C_c(\mathbb{R}^n)$ to the Riemann integral $\int_{\mathbb{R}^n} f \in \mathbb{C}$ is a Radon integral. The Lebesgue measure μ_n is defined to be the corresponding outer Radon measure on $\mathbb{R}^n$. Note that we have $\mu_n(gA) = |\det(g)|\mu_n(A)$ for any $g \in \mathrm{GL}_n(\mathbb{R})$ and $A \in \mathcal{B}(\mathbb{R}^n)$.

3.4. **Haar measure.** Let G be a locally compact Hausdorff topological group. In this section, we will define Haar measures on G and study some properties of this kind of measures.

Definition 33. *A Borel measure μ on G is* left-invariant *if $\mu(gS) = \mu(S)$ for all $g \in G$ and $S \in \mathcal{B}$. A* left Haar measure *on G, denoted $d_l g$, is a nonzero left-invariant outer Radon measure on G. Right Haar measure $d_r g$ is defined similarly.*

Remark 34. In terms of Radon integrals, the condition $\mu(gS) = \mu(S)$ for any measurable S is equivalent to

$$\int_G f(x) d\mu(x) = \int_G f(g^{-1}x) d\mu(x)$$

for any $f \in C_c(G)$. Indeed, it suffices to check this for $f = 1_S$ where $S \subset G$ is measurable.

Convention 35. To be more precise, a left Haar measure μ is a map from Borel sets of G to $[0, \infty]$. However, for convenience, we will usually denote a left Haar measure of G to be $d_l g$ and a right Haar measure by $d_r g$, where g is understood to be an element of G. For example, the left-invariant property is short-handed as $d_l(hg) = d_l(g)$, where $d_l(hg)$ is understood to be the measure of G obtained by pushforward $d_l g$ via left-multiplication by h^{-1}, i.e. $\Omega \mapsto \mu(h\Omega)$. Sometimes $d_l(hg)$ would cause ambiguity, where it could mean either pushing forward $d_l g$ via left-multiplication by h^{-1}, i.e. $\Omega \mapsto d_l(h\Omega)$, or pushing forward $d_l h$ via right-multiplication by g^{-1}, i.e. $\Omega \mapsto d_l(\Omega g)$, but we will try to be more precise when the situation arises. A better convention is $d(L_{h^{-1}}g)$ or $d(R_{g^{-1}}h)$.

Theorem 36 (Existence and uniqueness of Haar measure)**.** *Let G be a locally compact topological group. There exists a left Haar measure μ on G and every other left Haar measure on G is $c\mu$ for some $c \in \mathbb{R}_{>0}$.*

Example 37. On $\mathbb{R}^n$, the Lebesgue measure is a Haar measure.

Remark 38. A left Haar measure need not be right-invariant. For example, consider

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R} \right\}$$

then G has a left Haar measure given by $\mu_L(S) = \int_S \frac{1}{a^2} dadb$ and a right Haar measure given by $\mu_R(S) = \int_S \frac{1}{|a|} dadb$.

Let μ be a left Haar measure on G, then G is compact if and only if $\mu(G) < \infty$. The normalised Haar measure on G is the unique Haar measure μ such that $\mu(G) = 1$.

Example 39. Let k be a nonarchimedean local field with valuation ring $\mathcal{O}$. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$ and π be a uniformiser of $\mathcal{O}$. There is a Haar measure μ on k satisfying $\mu(\mathcal{O}) = 1$.

For example, we will show $\mu(\mathfrak{m}) = (\#\mathcal{O}/\mathfrak{m})^{-1}$ (here $\#\mathcal{O}/\mathfrak{m}$ refers to number of elements of this finite field). Indeed, as $\mathcal{O}_k$ is a disjoint union of $a + \mathfrak{m}$'s where $a \in \mathcal{O}/\mathfrak{m}$ and that μ is left-invariant, we find

$$1 = \mu(\mathcal{O}) = \sum_{a \in \mathcal{O}/\mathfrak{m}} \mu(a + \mathfrak{m}) = (\#\mathcal{O}/\mathfrak{m})\mu(\mathfrak{m}).$$

Similarly, one can show that $\mu(\pi^n\mathcal{O}) = (\#\mathcal{O}/\mathfrak{m})^{-n}$ for $n \in \mathbb{Z}$.

For another example of a computation with μ, we will show $\mu(aA) = |a|_k\mu(A)$ for any open A of k and $a \in k^\times$. Indeed, let $a = u\pi^n$ where $u \in \mathcal{O}^\times, n \in \mathbb{Z}$ and if $A = \mathcal{O}$ then

$$\mu(aA) = \mu(\pi^n\mathcal{O}) = (\#\mathcal{O}/\mathfrak{m})^{-n} = |a|_k\mu(A).$$

As $\pi^n\mathcal{O}$'s form a basis of open neighborhoods of $0 \in k$ so from the above computation, we are done. In fact, $\mu(aA) = |a|_k\mu(A)$ holds for any choice of Haar measure on k.

3.4.1. *Modular quasicharacter.* Let d_rg be a right Haar measure on G. Then $d_r(hg)$ is also a right Haar measure. Therefore, by uniqueness of right Haar measure, there exists a positive real $\delta_G(h)$ so $d_r(hg) = \delta_G(h)d_rg$. We define the *modular quasicharacter* to be the corresponding group homomorphism $\delta_G : G \to \mathbb{R}_{>0}$ [3]. Note that δ_G does not depend on the choice of a left/right Haar measure on G.

Proposition 40. *Let d_rg, d_lg be right, left Haar measures of G, respectively. Then the following are equivalent ways to define the modular quasicharacter:*

(a) $d_r(hg) = \delta_G(h)d_rg$ for all $h \in G$,
(b) $d_l(gh) = \delta_G(h)^{-1}d_lg$ for all $h \in G$,
(c) $d_r(g^{-1}) = \delta_G(g)^{-1}d_rg$,
(d) $d_l(g^{-1}) = \delta_G(g)d_lg$.

Furthermore, if we given d_rg, we can choose d_lg to be such that $d_lg = d_r(g^{-1})$, or equivalently, $d_rg = \delta_G(g)d_lg$.

Finally, every left Haar measure is right Haar measure if and only if $\delta_G \equiv 1$ on G. If this is the case, we say G is unimodular.

[3] Some authors define modular quasicharacter to be the multiplicative inverse of δ_G, such as in [Fol16]. Our choice for the definition of δ_G is reflected in Proposition 115

Sketch. We will prove (a) implies (c). Note that $d_r(g^{-1})$ is a left Haar measure. Next, we show $\delta_G(g)^{-1}d_rg$ is also a left Haar measure. We have

$$\begin{aligned}\int_G f(hg)\delta_G(g)^{-1}d_rg &= \int_G f(g)\delta_G(h^{-1}g)^{-1}d_r(h^{-1}g),\\ &= \int_G f(g)\delta_G(h^{-1}g)^{-1}\delta_G(h^{-1})d_rg,\\ &= \int_G f(g)\delta_G(g)^{-1}d_rg.\end{aligned}$$

By uniqueness of left Haar measure, we find $d_r(g^{-1}) = c\delta_G(g)^{-1}d_rg$ for some constant c. Changing g to g^{-1} (i.e. pushforward two measures under taking inversion, which should give us the same equality), we find

$$d_r(g) = c\delta_G(g)d_r(g^{-1}) = c^2\delta_G(g)\delta_G(g^{-1})d_r(g^{-1}) = c^2d_r(g^{-1}),$$

hence $c = 1$.

To show (c) implies (b). As $d_r(g^{-1})$ is a left Haar measure so we have $d_l(g) = cd_r(g^{-1})$ for some $c \in \mathbb{R}_{>0}$. Then we have

$$d_l(gh) = cd_r(h^{-1}g^{-1}) = \delta_G(h^{-1})cd_r(g^{-1}) = \delta_G(h)^{-1}d_lg.$$

The other equivalences of (a), (b), (c), (d) can be done similarly.

Next, we show $d_rg = \delta_G(g)d_l(g)$ implies $d_lg = d_r(g^{-1})$. Indeed, by (c) and (d), we find $d_r(g^{-1}) = \delta_G(g)^{-1}d_r(g) = d_l(g)$.

Finally, we show if every left Haar measure is right Haar measure then $\delta_G = 1$. Let $d_r = cd_l$, then from (a), as we fix h, we find

$$c\delta_G(h)d_lg = \delta_G(h)d_rg = d_r(hg) = c^{-1}d_l(hg) = c^{-1}d_lg.$$

This follows $\delta_G(h) = c^{-2}$, a constant. As δ_G is a group homomorphism, we find $\delta_G \equiv 1$. ☐

3.4.2. *Haar measure on a homogeneous space.* In this section, let G be a locally compact group with closed subgroup H. Then G acts on G/H by left-multiplication. We say a measure μ on G/H is *G-invariant* if $\mu(A) = \mu(xA)$ for any $x \in G$ and measurable $A \subset G/H$.

Theorem 41. *Let H be a closed subgroup of G with corresponding modular quasicharacters δ_H, δ_G. A necessary and sufficient condition for G/H to have nonzero G-invariant Borel measure $\mu_{G/H}$ is that the restriction to H of δ_G equals δ_H. In this case, such a measure is unique up to positive scalar, and it can be normalised so that for any $f \in C_c(G)$, we have*

$$\int_{G/H} f^H d\mu_{G/H} = \int_G f d\mu_G$$

where $f^H \in C_c(G/H)$ is defined by

$$f^H(x) = \int_H f(xh)d\mu_H.$$

Sketch. We sketch the proof when G, H are unimodular. We denote the projection $p : G \to G/H$.

In fact, the map $C_c(G) \to C_c(G/H)$ sending $f \mapsto f^H$ is onto, which we will not prove here but refer to [Fol16, p.62]. To show $\mu_{G/H}$ can be defined as in the theorem, we need to show $f^H \mapsto \int_G f d\mu_G$ is a well-defined G-invariant positive linear functional on $C_c(G/H)$. By surjectivity of $C_c(G) \to C_c(G/H)$, it suffices to show that if $f \in C_c(G)$ and $f^H = 0$ then $\int_G f d\mu_G = 0$. Let

$\varphi \in C_c(G/H)$ such that $\varphi = 1$ on $p(\text{supp } f)$, then there exists $g \in C_c(G)$ so $g^H = \varphi$. Assuming $f^H = 0$, we find

$$\begin{aligned} 0 &= \int_G g(x) f^H(x) dx = \int_G \int_H g(x) f(xh) dh dx, \\ &= \int_H \int_G g(x) f(xh) dx dh = \int_H \int_G g(xh) f(x) dx dh, \\ &= \int_G \int_H g(xh) f(x) dh dx = \int_G f(x) g^H(x) dx, \\ &= \int_G f(x) dx. \end{aligned}$$

We are done. □

Remark 42. One can consider the space of right cosets $H \backslash G$ and modify the above theorem accordingly.

In general, one can a define left G-invariant measure on a space under a continuous transitive action of G as follows.

Definition 43. *Let S be a locally compact topological space then S is a G-*space *if there is a continuous left action of G on S, i.e. a continuous map from $G \times S$ to S such that $s \mapsto xs$ is a homeomorphism of S, and $x(ys) = (xy)s$ for all $x, y \in G, s \in S$. A G-space is called transitive if for every $s, t \in S$ there exists $x \in G$ such that $xs = t$.*

If S is a transitive G-space then for any $s_0 \in S$, the isotropy/stabiliser group $H = \{x \in G : xs_0 = s_0\}$ of s_0 is a closed subgroup of G and $\phi : G \to S$ by $x \mapsto xs_0$ is a continuous surjection of G onto S. This induces a continuous bijection $\Phi : G/H \to S$ such that $\Phi \circ p = \phi$ where $p : G \to G/H$ is the quotient map. Note that it is generally not the case that Φ has continuous inverse. For example, consider $G = \mathbb{R}$ with the discrete topology, acting by translation on $S = \mathbb{R}$ with the usual topology. We call S a *homogeneous space* if Φ is a homeomorphism. With this, we can identify S with G/H and a G-invariant measure on G/H with a G-invariant measure on S.

3.4.3. *Haar measure from a fundamental domain.* When H is a discrete subgroup of G, one can determine $\mu_{G/H}$ by integrating with respect to μ_G over a fundamental domain F of G.

Definition 44. *Given a locally compact topological group G and a discrete subgroup H, a measurable set $F \subset G$ is a* strict fundamental domain *for $H \setminus G$ if the projection $\pi : F \to H \setminus G$ is a bijection. A measurable set $F \subset G$ is a* fundamental domain *for $H \setminus G$ if F differs from a strict fundamental domain by a set of Haar measure 0.*

When we have such a fundamental domain F, we can define a G-invariant measure on $H \setminus G$ by integrating over F.

Proposition 45. *Let G be a locally compact topological group with a left Haar measure $d\mu_G$, let H be a countable discrete subgroup of G, and let $F \subset G$ be a fundamental domain for $H \setminus G$. Then the quotient measure $H \setminus G$ can be given by*

$$\int_{H \backslash G} f(Hg) d\mu_{H \backslash G}(Hg) = \int_F f(g) d\mu_G(g).$$

Proof. By uniqueness of G-invariant measure on $H \setminus G$, it suffices to check that

$$\int_G f(g) dg = \int_F \sum_{h \in H} f(hg) dg$$

for all $f \in C_c(G)$. As $G = \bigcup_{h\in H} hF$, we find

$$\int_G f(g)dg = \sum_{h\in H}\int_{hF} f(g)dg = \sum_{h\in H}\int_F f(hg)dg = \int_F \sum_{h\in H} f(hg)dg$$

by left-invariance of G and the fact that we can exchange the sum and the integral by Fubini's theorem. □

3.4.4. *Haar measure from closed subgroups.* We have the following useful proposition that describes a Haar measure on $G = ST$ in terms of Haar measures on its closed subgroups S and T.

Proposition 46 (Theorem 8.32 of [Kna02])**.** *Suppose S and T are closed subgroups of G with compact intersection and the product map $S \times T \to G$ is open with image exhausting G except possibly for a set of Haar measure 0. Then one can normalise the left and right Haar measures on S and T, respectively, so that*

$$\int_G f(g)d_lg = \int_{S\times T} f(st)\frac{\delta_T(t)}{\delta_G(t)}d_lsd_rt = \int_{S\times T}\frac{f(st)}{\delta_G(t)}d_lsd_rt.$$

In particular, if G is unimodular, then

$$\int_G f(g)dg = \int_{S\times T} f(st)d_lsd_rt.$$

Proof. The group $S \times T$ acts continuously on $ST \subset G$ by $(s,t)w = swt^{-1}$, and the isotropy group at 1 is $K \times K$ where $K = S \cap T$. Thus, we have a bijective continuous map $\Phi : (S \times T)/(K \times K) \to ST$ sending $(s,t) \mapsto st^{-1}$. This map is a homeomorphism (i.e. has continuous inverse) since multiplication $S \times T \to G$ is an open map. Hence, a left Haar measure d_lg of G restricts to a Borel measure on ST, and hence obtaining a Borel measure $d\mu$ on $(S \times T)/(K \times K)$ via change of variables formula Proposition 28 for measures:

$$\int_{(S\times T)/(K\times K)} f(\Phi(s,t))d\mu = \int_{(S\times T)/(K\times K)} f(st^{-1})d\mu = \int_{ST} f(g)d_lg.$$

We denote $L_g, R_g : G \to G$ to be left/right translation maps. From Proposition 40, we have $d_l(L_{s_0}R_{t_0^{-1}}g) = \delta_G(t_0)d_lg$, which gives

$$\begin{aligned}\int_{(S\times T)/(K\times K)} f(s,t)d\mu(L_{(s_0,t_0)}(s,t)) &= \int_{(S\times T)/(K\times K)} f(s_0^{-1}s, t_0^{-1}t)d\mu,\\ &= \int_{(S\times T)/(K\times K)} (f\circ\Phi^{-1}\circ L_{s_0}R_{t_0}^{-1}\circ\Phi)(s,t)d\mu,\\ &= \int_{ST}(f\circ\Phi^{-1})(L_{s_0^{-1}}R_{t_0}g)d_lg,\\ &= \int_{ST}(f\circ\Phi^{-1})(s_0^{-1}gt_0)d_lg,\\ &= \int_{ST}(f\circ\Phi^{-1})(g)d_l(s_0gt_0^{-1}),\\ &= \int_{ST}(f\circ\Phi^{-1})(g)\delta_G(t_0)d_l(g),\\ &= \delta_G(t_0)\int_{(S\times T)/(K\times K)} f(s,t)d\mu,\end{aligned}$$

or in our convention,

$$d\mu(L_{(s_0,t_0)}x) = \delta_G(t_0)d\mu(x) \tag{1}$$

on $(S \times T)/(K \times K)$. We define measure $d\tilde{\mu}(s,t)$ on $S \times T$ by

$$\int_{S\times T} f(s,t)d\tilde{\mu}(s,t) = \int_{(S\times T)/(K\times K)} \left[\int_K f(sk,tk)dk\right] d\mu((s,t)K),$$

where dk is Haar measure on compact K normalised to have volume 1. From (1), we have $d\tilde{\mu}(s_0 s, t_0 t) = \delta_G(t_0)d\tilde{\mu}(s,t)$. Note that $\delta_G(t)d_l s d_r t$ also satisfies this condition. Therefore, $d\tilde{\mu}(s,t) = \delta_G(t)d_l s d_r t$ for suitable normalisation of $d_l s d_r t$ (to see this, mimic the proof that a left Haar measure is unique up to scalar, see [Kna02, Theorem 8.23]). Hence, we find

$$\int_{ST} f(g)d_l g = \int_{S\times T} f(st^{-1})\delta_G(t)d_l s d_r t$$

for all $f \in C_c(ST)$. Changing t by t^{-1} on the right hand side via Proposition 40 and replacing ST by G on the left hand side, we are done. □

3.4.5. *Haar measure on restricted product.* In this section, we will construct certain Haar measure on restricted products, which will be required later in defining Haar measure of adelic points of linear algebraic groups. We first define restricted products of a family of topological spaces.

Definition 47. *Let (X_i) be a family of topological spaces indexed by $i \in I$, and let (U_i) be a family of open sets $U_i \subset X_i$. The* restricted product $\prod'_{i\in I} X_i$ with respect to U_i's *is the topological space*

$$X = \prod_{i\in I}{}'(X_i, U_i) := \left\{(x_i) \in \prod X_i : x_i \in U \text{ for almost all } i \in I\right\}.$$

with the basis of open sets

$$\left\{\prod V_i : V_i \subset X_i \text{ is open for all } i \in I and V_i = U_i \text{ for almost all } i\right\},$$

where almost all means all but finitely many.

Remark 48. We refer to [Sut19] for the proofs of the following remarks about restricted products:

(1) In general, the restricted product X is not the subspace topology from $\prod X_i$ as the former has more open sets [4].
(2) For a finite set $S \subset I$ then by letting

$$X_S := \prod_{i\in S} X_i \times \prod_{i\notin S} U_i$$

then X_S is an open set of X whose subspace topology is precisely the product topology of the X_i's and U_i's. As $\prod' X_i = \bigcup_S X_S$ over all finite sets $S \subset I$, this gives another way to define the restricted product as the direct limit of X_S's.
(3) If X_i's are locally compact and almost all of the U_i's are compact then the restricted product $\prod' X_i$ is locally compact.

Proposition 49 (p. 185 of [VR99])**.** *Let $G = \prod_{v\in J} G_v$ be the restricted direct product of locally compact groups G_v with respect to family of compact subgroups $H_v \subset G_v$ (except for some finite set of places J_∞). Let μ_v be a left Haar measure on G_v normalised so that $\prod_{v\notin J_\infty} \mu_v(H_v)$ converges. Then there is a unique Haar measure μ on G such that for each finite set of indices S containing J_∞, the restriction μ_S of μ to*

$$G_S = \prod_{v\in S} G_v \times \prod_{v\notin S} H_v$$

is the product measure.

[4] recall the product topology on $\prod_i V_i$ of topological spaces V_i is the coarsest topology for which all the projections are continuous

Proof. The finiteness of $\prod_{v\nmid\infty}\int_{H_v} dg_v$ guarantees that the product measure μ_S on G_S is a Haar measure, i.e. is finite on compact subsets $\prod_{v\nmid\infty} H_v$.

Next, we will show the existence of such a Haar measure on G. As G is locally compact, we can choose a left Haar measure μ on G such that for some fixed finite set of S of indices containing J_∞, the restriction of μ to G_S is the product measure μ_S. This measure μ is independent of the choice of S because if we consider another finite set S' of indices containing J_∞, again because of uniqueness of Haar measure on $G_{S\cup S'}$ whose restriction to G_S is μ_S, μ restricted to $G_{S\cup S'}$ must also be a product measure. Hence, μ restricted to $G_{S'}$ is also a product measure. $\square$

4. Analytic manifolds and integrations

Over a complete valued field k with respect to a nontrivial absolute value, one can develop a theory of k-analytic functions and k-analytic manifolds that closely resembles the classical setting of real analytic functions and real analytic manifolds. Furthermore, when k is a local field, one can also define integration of differential forms on k-analytic manifolds. In this section, we will describe this process, following [Igu00].

On a different note, unlike most references we find about differential geometry, we will discuss manifolds and its related objects in the language of sheaf theory. One reason is that this language is also used in algebraic geometry, so in our naive view, it seems to be a more universal language than describing manifolds via compatible charts. For example, such point of view is also taken in [Ram05], [Wed16].

4.1. **Analytic functions.** For every $a = (a_1, \ldots, a_d) \in k^d$ and every $r \in \mathbb{R}_{>0}$, we denote the closed and open polydisks of radius r centered at a in k^d to be

$$D(a, r) = \{x \in k^d : |x_i - a_i| \leq r \ \forall i\},$$
$$D_0(a, r) = \{x \in k^d : |x_i - a_i| < r \ \forall i\}.$$

We consider power series $f(T) = \sum_{n \in \mathbb{Z}^d_{\geq 0}} f_n T^n \in k[[T_1, \ldots, T_d]]$ of d variables with coefficients in k, where we denote $T^n := T_1^{n_1} \cdots T_d^{n_d}$ and $|n| = n_1 + \cdots + n_d$.

A power series $f = \sum_{n \in \mathbb{Z}^d_{\geq 0}} f_n T^n \in k[[T]]$ is said to be *convergent* if its radius of convergence, defined by $\rho(f) = \left(\limsup_{|n| \to \infty} |f_n|^{1/|n|}\right)^{-1}$ is positive. We have that for any $0 < r < \rho(f)$, the series $\sum_n f_n T^n$ converges in k for all $T \in D_0(0, r)$.

Let U be an open subset of k^d. We say a function $f : U \to k$ is *k-analytic* in U if for each $a \in U$, there is a real number $r > 0$ and a convergent power series $f_a \in k[[T]]$ such that $f(x) = f_a(x - a)$ for all $x \in D(a, r) \subset U$. Every k-analytic function is continuous. If a k-analytic function on U does not vanish anywhere, then its inverse is k-analytic as well.

For a positive integer m, a function $f : U \to k^m$ defined by $u \mapsto (f_1(u), \ldots, f_m(u))$ is k-analytic if each f_i is analytic for $1 \leq i \leq n$. Composition of k-analytic functions is k-analytic.

For a k-analytic function $f : U \to k$ on an open set U of k^d, one can define its partial derivatives at $a \in U$ to be

$$\frac{\partial f}{\partial x_i}(a) := \lim_{t \to 0} \frac{f(a + t\varepsilon_i) - f(a)}{t},$$

for $i \in \{1, \ldots, d\}$, where $\varepsilon_i = (0, \ldots, 1, \ldots, 0)$ which has 1 in the i-th place and 0 everywhere else. We also know that $\partial f / \partial x_i$'s are k-analytic. We define the Jacobian matrix of a k-analytic map $f : U \to k^d$ as $Df(a) = (\partial f_i / \partial x_j(a))$, where $f = (f_1, \ldots, f_d)$. The determinant of $Df(a)$ defines an analytic map J_f on U, called the *Jacobian determinant* of f.

The inverse function theorem and implicit function theorem also hold over any complete valued field k.

Theorem 50 (Inverse function theorem)**.** *Let $f : U \to k^d$ be a k-analytic function where U is an open subset of k^d. Let $a \in U$ be such that the Jacobian matrix $Df(a)$ of f at a does not vanish. Then there exist an open neighborhood U_a of a such that $f(U_a)$ is an open neighborhood of $f(a)$ in k^d and a k-analytic function $g : f(U_a) \to U_a$ such that $g \circ f = id_{U_a}$ and $f \circ g = id_{f(U_a)}$.*

Theorem 51 (Implicit function theorem)**.** *Let $F = (F_1, \ldots, F_m)$ where $F_1, \ldots, F_m \in k[[x_1, \ldots, x_n, y_1, \ldots, y_m]]$ are k-analytic functions on a neighborhood of $(0, 0)$ such that $F_i(x, y) = 0$ for all $1 \leq i \leq m$. If $\det(\partial F_i / \partial y_j(0, 0)) \neq 0$ then there exist k-analytic functions $f_1, \ldots, f_m \in k[[x_1, \ldots, x_n]]$ on some open neighborhood U of $0 \in k^n$ such that for $f = (f_1, \ldots, f_m)$, $f(0) = 0$ and $F(x, f(x)) = 0$ for all $x \in U$.*

For the proofs of these two theorems, we refer to [Igu00, Section 2.1].

4.2. **Locally ringed space.** In this section, we would like to introduce the notion of locally ringed spaces, which we will later use to define a k-analytic manifold.

Definition 52. *A* ringed space *is a pair* $(X, \mathcal{O}_X)$ *consisting of a topological space* X *and a sheaf of rings* $\mathcal{O}_X$*. A* k-ringed space *is a ringed space* $(X, \mathcal{O}_X)$ *where* $\mathcal{O}_X$ *is a sheaf of* k*-algebras. A morphism of ringed spaces* $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *is given by a continuous map* $f : X \to Y$ *and a morphism of sheaves* $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ *over* Y*.*

A locally ringed space $(X, \mathcal{O}_X)$ *is a ringed space* $(X, \mathcal{O}_X)$ *whose stalks are local rings. Given the stalk* $\mathcal{O}_{X,x}$ *at* x *with its unique maximal ideal* $\mathfrak{m}_x$*, the* residue field of X at x *is* $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$*. A morphism of locally ringed spaces* $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *is a morphism of ringed spaces such that the induced ring map* $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ *is a local ring map.*

We say a ringed space $(X, \mathcal{O}_X)$ *is* locally isomorphic *to* $(Y, \mathcal{O}_Y)$ *if for each* $x \in X$*, there exists an open neighborhood* U *of* x *and an isomorphism* $(U, \mathcal{O}_X|_U) \cong (V, \mathcal{O}_Y|_V)$ *of sheaves where* V *is some open subset of* Y*.*

Remark 53. For a locally ringed space $(X, \mathcal{O}_X)$, given $f \in \mathcal{O}_X(U)$, we can talk about the *value of* f *at* $x \in U$ as the image of f in $\kappa(x)$. Hence, one would like to think of sections of $\mathcal{O}_X$ as functions on X.

Example 54. Let M be a real C^∞-manifold. Then we can define a structure sheaf $\mathcal{O}_M$ for M where $\mathcal{O}_M(U)$ is the ring of smooth functions $f : U \to \mathbb{R}$. $(M, \mathcal{O}_M)$ is then a locally $\mathbb{R}$-ringed space, as for $x \in M$, $\mathcal{O}_{M,x}$ is the ring of germs of smooth functions at x, which is a local ring with maximal ideal being functions that vanish at x. The value of $f \in \mathcal{O}_M(U)$ at $x \in U$, by definition above, is precisely $f(x)$. Furthermore, $(M, \mathcal{O}_M)$ is locally isomorphic to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ with its sheaf of C^∞-functions. Indeed, for any $x \in M$, we can choose a chart $(U, \varphi : U \to \mathbb{R}^n)$ of x, then $(U, \mathcal{O}_M|_U)$ is isomorphic to $(\varphi(U), \mathcal{O}_{\mathbb{R}^n}|_{\varphi(U)})$ by sending a smooth function $f : U \to \mathbb{R}$ on U to a smooth function $f \circ \varphi^{-1}$ on $\varphi(U) \subset \mathbb{R}^n$.

Combined with the previous example, the following proposition indicates that saying M is a real C^∞-manifold is the same as saying that M is a $\mathbb{R}$-ringed space that is locally isomorphic to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ with its sheaf of C^∞-functions.

Theorem 55. *Let* $(M, \mathcal{O}_M)$ *be a* $\mathbb{R}$*-ringed space that is locally isomorphic to* $(\mathbb{R}, \mathcal{O}_{\mathbb{R}^d})$ *with its sheaf of* C^∞*-functions. Then* M *can be equipped with a structure of a real* C^∞*-manifold, with* $\mathcal{O}_M$ *being the sheaf of smooth functions on* M*.*

Proof. We can cover M by open sets U's such that for each U, there is an isomorphism $\varphi_U : (U, \mathcal{O}_M|_U) \xrightarrow{\sim} (V, \mathcal{O}_{\mathbb{R}^d}|_V)$, where V is open in $\mathbb{R}^d$. We say (U, φ_U) is a *chart* of M. The $\mathbb{R}$-algebra of $\mathbb{R}$-analytic functions on U is $\mathcal{O}_M(U)$. If we are given another chart $(U', \varphi_{U'})$ for M, we have an isomorphism of locally $\mathbb{R}$-ringed spaces

$$\varphi_{U'}|_{U\cap U'} \circ \varphi_U^{-1}|_{\varphi^{-1}(U\cap U')} : (\varphi_U(U \cap U'), \mathcal{O}_{\mathbb{R}^d}|_{\varphi_U(U\cap U')}) \to (\varphi_{U'}(U \cap U'), \mathcal{O}_{\mathbb{R}^d}|_{\varphi_{U'}(U\cap U')}).$$

The following lemma implies that the above morphism is precisely the chart-compatibility condition in the classical definition of manifolds via charts and atlas.

Lemma 56. *Let* $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ *be the sheaf of* C^∞*-functions on* $\mathbb{R}^n$*. Let* $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ *be open subsets with the induced structures of locally* $\mathbb{R}$*-ringed spaces* $\mathcal{O}_U, \mathcal{O}_V$ *from* $\mathbb{R}^n, \mathbb{R}^m$*, respectively. Then every morphism* $f : (U, \mathcal{O}_U) \to (U, \mathcal{O}_V)$ *of locally* $\mathbb{R}$*-ringed spaces is* k*-analytic. Furthermore, the morphism of sheaves is given by sending* $g \in \mathcal{O}_V(V')$ *to* $g \circ f \in \mathcal{O}_U(f^{-1}(V'))$ *for any open subset* V' *of* V*.*

Conversely, any $\mathbb{R}$*-analytic map* $f : U \to V$ *induces a morphism of locally* $\mathbb{R}$*-ringed spaces via taking compositions.*

Sketch of proof of lemma. Let V' be an open subset in V and $a \in f^{-1}(V')$. We have the following commutative diagram

$$\begin{array}{ccccccc}
\mathcal{O}_V(V') & \longrightarrow & \mathcal{O}_{V,f(a)} & \longrightarrow & \mathcal{O}_{V,f(a)}/\mathfrak{m}_{V,f(a)} & \xrightarrow{\sim} & \mathbb{R} \\
\downarrow{\scriptstyle f(V')} & & \downarrow{\scriptstyle f_a} & & \downarrow{\scriptstyle \overline{f_a}} & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathcal{O}_U(f^{-1}(V')) & \longrightarrow & \mathcal{O}_{U,a} & \longrightarrow & \mathcal{O}_{U,a}/\mathfrak{m}_{U,a} & \xrightarrow{\sim} & \mathbb{R}
\end{array}$$

In this diagram, the first row corresponds to the evaluation of elements in $\mathcal{O}_V(V')$ at $f(a)$ and similarly for the second row. We have $f_a : \mathcal{O}_{V,f(a)} \to \mathcal{O}_{U,a}$ is a local ring map so it induces $\overline{f_a}$ which corresponds to the identity map on $\mathbb{R}$ because $f(V')$ is a morphism of $\mathbb{R}$-algebras. The commutativity of the diagram implies that $f(V')(g)(a) = g(f(a))$ for $g \in \mathcal{O}_V(V')$, as desired. □

We are done. □

Example 57. Let A be a commutative ring with unity. Let $X = \operatorname{Spec} A$ to be the set of all prime ideals of A. In this example, we show that X can be equipped with a structure of a ringed space. X is then called an affine scheme.

First, X is a topological space with closed sets being $V(S) = \{\mathfrak{p} \in \operatorname{Spec} A : S \subset \mathfrak{p}\}$ for all subsets S of A. One can also show that X has basis of open sets $D(f) = \{\mathfrak{p} \in \operatorname{Spec} A : f \notin \mathfrak{p}\}$ where $f \in A$.

To define the structure sheaf $\mathcal{O}_X$ of X, it suffices to define this on the basis of open sets of X, i.e. we let $\mathcal{O}_X(D(f)) = A_f$, the localisation of A at the set $\{f, f^2, \ldots\}$.

In this case, for $f \in \mathcal{O}_X(X) = A$, the value of f at $\mathfrak{p} \in X$ is the image of f in $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$, which is $f \pmod{\mathfrak{p}}$. A scheme is a ringed space that is locally isomorphic to affine schemes.

4.3. **Analytic manifolds.** Theorem 55 suggests that we define k-analytic manifolds as follows.

Definition 58. *A k-analytic manifold of dimension d is a k-ringed space $(M, \mathcal{O}_M)$ which is locally isomorphic to $(k^d, \mathcal{O}_{k^d})$ with its sheaf of k-analytic functions. This follows that $(M, \mathcal{O}_M)$ is a locally k-ringed space. A morphism $\phi : M \to N$ of two k-analytic manifolds is a morphism of locally k-ringed spaces.*

Remark 59. With the same argument as in Theorem 55, one can show that our definition of k-analytic manifolds is the same as the definition of k-analytic manifolds via charts and atlas.

Remark 60. In most situations, M is assumed to be paracompact and Hausdorff. For example, these conditions give the existence of a continuous partition of unity on coverings of M (see [Cra11]), and we will later use this to define integration of top-forms on M.

Next, we will define (co)tangent bundles/vectors of a k-analytic manifolds as derivations. In fact, the following definitions work for any locally k-ringed space $(M, \mathcal{O}_M)$, but we will restrict our attention to M being a k-analytic manifold.

Definition 61 (Tangent bundle)**.** *Let $(M, \mathcal{O}_M)$ be a k-analytic manifold. A k-derivation of $\mathcal{O}_M$ is a k-linear homomorphism $D : \mathcal{O}_M \to \mathcal{O}_M$ of sheaves such that $D_U(fg) = fD_U(g) + gD_U(f)$ for all $U \subset M$ open, $f, g \in \mathcal{O}_M(U)$. Denote by $\operatorname{Der}_k(\mathcal{O}_M)$ the k-vector space of k-derivations of $\mathcal{O}_M$. It is also an $\mathcal{O}_M(M)$-module via*

$$(g \cdot D)_U(f) := g_U D_U(f),\ D \in \operatorname{Der}_k(\mathcal{O}_M), g \in \mathcal{O}_M(M), f \in \mathcal{O}_M(U).$$

We define the tangent bundle *TM to be the sheaf of $\mathcal{O}_M$-modules via $TM(U) := \operatorname{Der}_k(\mathcal{O}_M|_U)$. A section of tangent bundle over U is called a* vector field. *The* tangent space T_pM of M at p *is the stalk of TM at p, which is a k-vector space of k-derivations* $\operatorname{Der}_k(\mathcal{O}_{M,p})$. *Equivalently, by composing with $\mathcal{O}_{M,p} \to \kappa(p) = \mathcal{O}_{M,p}/\mathfrak{m}_p \cong k$, we can describe T_pM as the k-vector space of k-derivations $\mathcal{O}_{M,p} \to k$ at p, i.e. $D \in T_pM$ then $D : \mathcal{O}_{M,p} \to k$ such that $D(fg) = f(a)D(g) + g(a)D(f)$.*

Remark 62. Let (x, U) be a chart of a k-analytic manifold M with coordinate functions $x_1, \ldots, x_d$ then $\frac{\partial}{\partial x_i} : \mathcal{O}_M|_U \to \mathcal{O}_M|_U$ sends $f \in \mathcal{O}_M(V)$ to $\frac{\partial f}{\partial x_i} \in \mathcal{O}_M(V)$ where $V \subset U$ is open. Here $\frac{\partial f}{\partial x_i} \in \mathcal{O}_M(V)$ is a k-valued function on V, sending $p \in V$ to $\frac{\partial f \circ x^{-1}}{\partial x_i}(x(p))$. And $\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\right)$ form a basis of the free $\mathcal{O}_M(U)$-module $\mathrm{Der}_k(\mathcal{O}_M|_U)$.

Definition 63. *The* cotangent bundle Ω^1_M *of an analytic manifold* $(M, \mathcal{O}_M)$ *is the sheaf of* $\mathcal{O}_M$*-modules* $\mathcal{H}om(TM, \mathcal{O}_M)$*. Concretely, a section over* U *is a morphism of sheaves* $f : TM|_U \to \mathcal{O}_M|_U$*, called a* differential 1-form over U*. Furthermore, we can define* $\Omega^p_M = \bigwedge^p \Omega^1_M$*, whose section over* $U \subset M$ *is called a* differential p-form over U*.*

We define a morphism $d : \mathcal{O}_M \to \Omega^1_M$ *of sheaves of* k*-vector spaces as follows:*

$$\begin{aligned} d : \mathcal{O}_M &\to \Omega^1_M, \\ f \in \mathcal{O}_M(U) &\mapsto (df : D \in TM_U = \mathrm{Der}_k(\mathcal{O}_M|_U) \mapsto D(f) \in \mathcal{O}_M|_U). \end{aligned}$$

In particular, we have $d(fg) = fdg + gdf$*.*

Remark 64. Let (x, U) be a chart of M with coordinate functions $x_1, \ldots, x_d : U \to k$. Then $dx_i \in \Omega^1_M(U)$ and $(dx_1, \ldots, dx_d)$ is a basis of $\Omega^1_M(U)$. This basis is dual to the basis $\left(\frac{\partial}{\partial x_i}\right)$ of $TM(U)$. For $f \in \mathcal{O}_M(U)$ then

$$df = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i} dx_i.$$

If $r \geq 1$ then $\Omega^i_M|_U$ is a free $\mathcal{O}_M|_U$-module with basis

$$dx_{i_1} \wedge \cdots \wedge dx_{i_r},\ 1 \leq i_1 < \cdots < i_r \leq d.$$

Remark 65. The cotangent bundle satisfies the following universal property: it is a sheaf of $\mathcal{O}_M$-modules equipped with differential $d : \mathcal{O}_M \to \Omega^1_M$, i.e. a morphism of sheaves of k-vector spaces satisfying $d(fg) = fdg + gdf$ where $f, g \in \mathcal{O}_M(U)$, that is universal among sheaves of $\mathcal{O}_M$-modules X equipped with differential $d : \mathcal{O}_M \to X$. The universal property implies that given a chart (U, x) M with coordinate functions $x_1, \ldots, x_d$, $\Omega^1_M(U)$ is a free $\mathcal{O}_M(U)$-module with basis dx_i.

A morphism of k-analytic manifolds $\phi : (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ will induce a morphism $\phi^* : \Omega^1_N \to \Omega^1_M$ of $\mathcal{O}_M$-modules. Concretely, if $f \in \mathcal{O}_N(U)$ then $d_N f \in \Omega^1_N(U)$ is sent to $d_M(f \circ \phi)$ where $f \circ \phi \in \mathcal{O}_M(f^{-1}(U))$.

Remark 66. In terms of coordinates, let $f : M \to N$ be a morphism of k-analytic manifolds, and let $(V, y), (U, x)$ be charts of M, N respectively with coordinate functions $x_1, \ldots, x_d$ for x and $y_1, \ldots, y_e$ for y. Then $\omega \in \Omega^p_N(U)$ can be written as

$$\omega = \sum_{1 \leq i_1 < \cdots < i_p \leq d} \omega_I dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

where $I = (i_1, \ldots, i_p)$ and the ω_I's are k-analytic functions on U. The morphism $f : M \to N$ of k-analytic manifolds will induce a differential p-form $f^*\omega \in \Omega^p_M(f^{-1}(U) \cap V)$ on $f^{-1}(U) \cap V \subset M$ defined by

$$\begin{aligned} f^*\omega &= \sum_I (\omega_I \circ \phi) d(x_i \circ \phi), \\ &= \sum_{1 \leq i_1 < \cdots < i_p \leq d} \sum_{1 \leq j_1 < \cdots < j_p \leq e} (\omega_I \circ \phi) \det \left(\frac{\partial x_{i_m} \circ \phi}{\partial y_{j_n}} \right)_{1 \leq m \leq d, 1 \leq n \leq e} dy_{j_1} \wedge \cdots \wedge dy_{j_p}. \end{aligned}$$

4.4. **Integration of differential forms.** In this section, we assume that k is a local field with a Haar measure μ. Let M be a k-analytic manifold of dimension d and let ω be a global differential d-form on M. We will define a measure on M by defining integration of the d-form ω [5].

First, we consider the case when $M = k^d$. We then can form a product measure $d\mu$ on M from the given Haar measure on the local field k. Suppose that over an open U of M, ω can be written as $h(x)dx_1 \wedge \cdots \wedge dx_d$ where h is a k-analytic function on U. With this, we define the measure $|\omega|$ on U to be

$$\int_U \varphi|\omega| := \int_U \varphi(x)|h(x)|_k d\mu$$

for any complex-valued $\varphi \in C_c(M)$ with compact support in U. To see what happen to this measure under k-bianalytic map $f : V \to V$, we first need the change of variables formula for k^d:

Theorem 67. *Let U be open set in k^d and $f : U \to k^d$ be an injective k-analytic map whose Jacobian J_f does not vanish on U. Then for any measurable positive (resp. integrable) function $\varphi : f(U) \to \mathbb{R}$, we have*

$$\int_{f(U)} \varphi(y)d\mu(y) = \int_U \varphi(f(x))|J_f(x)|_k d\mu(x).$$

Proof. We refer to [Igu00, Theorem 7.4.1] for the proof when k is a nonarchimedean local field. □

Now, consider a k-bianalytic map $f : V \to U$, where $U, V \subset k^d$ are open with coordinates $x_1, \ldots, x_d$ on U and $y_1, \ldots, y_d$ on V. Then $J_f(x) = \det(\partial(x_i \circ f)/\partial y_j)$. As

$$f^*\omega = h(f(x))J_f(x)dy_1 \wedge \cdots \wedge dy_d,$$

we have, by the change of variable formula

$$\begin{aligned}\int_V (\varphi \circ f)|f^*\omega| &= \int_V (\varphi \circ f)|h(f(x))|_k |J_f(x)|_k dy_1 \wedge \cdots \wedge dy_d,\\ &= \int_U \varphi|h(x)|_k dx_1 \wedge \cdots \wedge dx_d,\\ &= \int_U \varphi|\omega|.\end{aligned}$$

Next, we consider the case when M is any k-analytic manifold of dimension d.

Proposition 68. *There exists a unique measure $|\omega|$ such that for every chart (U, f) of M and every measurable positive (resp. integrable) function φ supported in U,*

$$\int_M \varphi|\omega| = \int_{f(U)} (\varphi \circ f^{-1})|(f^{-1})^*\omega|.$$

Sketch. To construct ω, by Riesz's representation theorem, it suffices to do this for φ with compact support. One can consider charts (U_i, f_i) of M covering support of φ and consider continuous partition of unity subordinated for these charts, i.e. a family (λ_i) of continuous real-valued functions on M such that supp $\lambda_i \subset U_i$ and $\sum \lambda_i = 1$ on supp φ. We then have

$$\int_M \varphi|\omega| := \sum_i \int_{f_i(U_i)} (\lambda_i \circ f_i^{-1})(\varphi \circ f_i^{-1})|(f_i^{-1})^*\omega|.$$

We show the independence of charts. Suppose we have another chart (U, g) with same $U \subset M$. Then $f \circ g^{-1} : g(U) \to f(U)$ is k-bianalytic map on k^d, so

$$\int_{f(U)} |(f^{-1})^*\omega| = \int_{g(U)} |(f \circ g^{-1})^*(f^{-1})^*\omega| = \int_{g(U)} |(g^{-1})^*\omega|.$$

[5] or to be more precise, we are integrating a density $|\omega|$

It is not difficult to show that our definition does not depend on the choice of a partition of unity of M. $\square$

5. Adeles

In this section, we will define the ring of adeles $\mathbb{A}_k$ associated to a global field k and study its topology. We then describe a functorial way to give a topology on $G(\mathbb{A}_k)$ for any linear algebraic group G. We also show that $G(k_v)$ is a k_v-analytic manifold for smooth G.

5.1. **Adeles of $\mathbb{Q}$.** We will review the construction of adeles $\mathbb{A}$ for $k = \mathbb{Q}$. Let S be always a finite nonempty set of places of $\mathbb{Q}$, including the infinite place. For convenience, we sometimes refer to $\mathbb{R}$ as $\mathbb{Q}_\infty$.

Definition 69. *The* adeles $\mathbb{A} = \mathbb{A}_\mathbb{Q}$ *of* $\mathbb{Q}$ *is the restricted product of the locally compact spaces* $\mathbb{Q}_p$ *with respect to the compact open subspace* $\mathbb{Z}_p$ *of* $\mathbb{Q}_p$. *In other words,* $\mathbb{A}$ *is a topological space whose elements are*

$$\mathbb{A}_\mathbb{Q} = \prod_{p \le \infty}{}' \mathbb{Q}_p := \left\{ (a_p)_p \in \prod_{p \le \infty} \mathbb{Q}_p : a_p \in \mathbb{Z}_p \text{ for almost all } p \right\},$$

here "almost all" means "all but finitely many". $\mathbb{A}$ *has a basis of open sets given by*

$$U_S \times \prod_{v \notin S} \mathbb{Z}_v,$$

where S *is a finite set of places of* $\mathbb{Q}$*, and* U_S *is an open set of*

$$\mathbb{Q}_S := \prod_{v \in S} \mathbb{Q}_v$$

under the product topology.

Let

$$\mathbb{Q}_S = \prod_{v \in S} \mathbb{Q}_v, \quad \widehat{\mathbb{Z}}^S = \prod_{v \notin S} \mathbb{Z}_v.$$

We find $\mathbb{Q}_S \times \widehat{\mathbb{Z}}^S$ is an open subring of $\mathbb{A}$ with the induced topology being the product topology [6]. Indeed, the open sets of $\mathbb{A}$ restricted to $\mathbb{Q}_S \times \widehat{\mathbb{Z}}^S$ are of the form $\prod_{v \in S} U_v \times \prod_{v \notin S} V_v$ where U_v is open in $\mathbb{Q}_v$, V_v is open in $\mathbb{Z}_v$ and $V_v = \mathbb{Z}_v$ for almost all $v \notin S$. This is precisely the open basis of the product topology of $\mathbb{A}_S := \mathbb{Q}_S \times \widehat{\mathbb{Z}}^S$. Furthermore,

$$\mathbb{A} = \bigcup_S \mathbb{Q}_S \times \widehat{\mathbb{Z}}^S$$

where S ranges over all finite sets of places of $\mathbb{Q}$. Also note that for $S \subset T$, we have an inclusion continuous map from $\mathbb{Q}_S \times \widehat{\mathbb{Z}}^S$ to $\mathbb{Q}_T \times \widehat{\mathbb{Z}}^T$ of topological rings. In other words, we find $\mathbb{A}_K = \varinjlim_S \mathbb{Q}_S \times \widehat{\mathbb{Z}}^S$. With this, $\mathbb{A}$ is a topological ring, under componentwise addition and multiplication.

Proposition 70. *The adeles ring* $\mathbb{A}$ *is a locally compact Hausdorff topological ring.*

Proof. We first show $\mathbb{A}$ is locally compact. Note that by Tychonoff's theorem, $\prod_{v \notin S} \mathbb{Z}_v$ is compact as each $\mathbb{Z}_v$ is compact. It follows that $\mathbb{Q}_S \times \widehat{\mathbb{Z}}^S$ is a finite product of locally compact spaces, hence locally compact. As each point $x \in \mathbb{A}$ lies in one of these spaces, we find $\mathbb{A}$ is locally compact.

Next, we show $\mathbb{A}$ is Hausdorff. Note that $\prod_v \mathbb{Q}_v$ is Hausdorff as $\mathbb{Q}_v$ is Hausdorff. It follows that $\mathbb{A}$ with the subspace topology of $\prod_v \mathbb{Q}_v$ is Hausdorff. In particular, for any two distinct points $x, y \in \mathbb{A}$, there exists two disjoint open sets $\prod_{v \in S} U_v \times \prod_{v \notin S} \mathbb{Q}_v$ and $\prod_{v \in T} V_v \times \prod_{v \notin T} \mathbb{Q}_v$ of $\prod_v \mathbb{Q}_v$ that contain x, y, respectively; here S, T are finite sets of places of $\mathbb{Q}$. Because $x, y \in \mathbb{A}$, one can enlarge S, T so that $\prod_{v \in S} U_v \times \prod_{v \notin S} \mathbb{Z}_v$ and $\prod_{v \in T} V_v \times \prod_{v \notin T} \mathbb{Z}_v$ are disjoint open sets of $\mathbb{A}$ containing x, y, respectively. Thus, we conclude that $\mathbb{A}$ is Hausdorff. □

[6] this notation of $\widehat{\mathbb{Z}}^S$ is motivated from the fact that it is the profinite completion of $\mathbb{Z}^S = \{x \in k | x \in \mathcal{O}_v \ \forall v \notin S\}$

For each place v of $\mathbb{Q}$, we have a continuous embedding

$$\mathbb{Q}_v \hookrightarrow \mathbb{A} : x_v \mapsto (0, \ldots, 0, x_v, 0, \ldots, 0).$$

Indeed, the preimage of a basis open set $\prod_{v \in S} U_v \times \prod_{v \notin S} \mathbb{Z}_v$ of $\mathbb{A}$ is either: $\varnothing$ if $0 \notin U_t$ for some place $t \in S, t \neq v$; or U_v if $0 \in U_t$ for all $t \in S \setminus \{v\}$ and $v \in S$; or $\mathbb{Z}_v$ if $0 \in U_t$ for all $t \in S$ and $v \notin S$.

We have a diagonal embedding

$$\mathbb{Q} \hookrightarrow \mathbb{A} : x \mapsto (x, x, \ldots, x).$$

This map is well-defined as $x \in \mathbb{Z}_v$ for almost all places v where $x \in \mathbb{Q}_v$. The image of $\mathbb{Q}$ under this embedding is called the *principal adeles*, which we will also denote $\mathbb{Q}$ for convenience.

Proposition 71. *$\mathbb{Q}$ is a discrete subgroup of $\mathbb{A}$.*

Proof. It suffices to show that $0 \in \mathbb{Q}$ has an open neighborhood U in $\mathbb{A}$ that does not intersect $\mathbb{Q} \setminus \{0\}$. Let $U = \{(x_v) \in \mathbb{A} : |x_v|_v < 1 \text{ if } v = \infty \text{ and } |x_v|_v \leq 1 \text{ if } v \neq \infty\}$ then U is open and $0 \in U$. By prime factorisation in $k = \mathbb{Q}$, we find $U \cap (\mathbb{Q} \setminus \{0\}) = \varnothing$, as desired. □

Let

$$\mathbb{A}^S = \prod_{v \notin S}{}' \mathbb{Q}_v.$$

Then we can identify $\mathbb{Q}_S \times \mathbb{A}^S$ with $\mathbb{A}$ via $\mathbb{Q}_S \times \mathbb{A}^S \hookrightarrow \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ where the latter map is addition on $\mathbb{A}$. It follows that $\mathbb{Q}_S \times \mathbb{A}^S$ is isomorphic to $\mathbb{A}$ as topological rings, with the product topology on $\mathbb{Q}_S$ and the restricted product topology on $\mathbb{A}^S$.

5.1.1. *Approximation theorem for adeles.*

Theorem 72. *For any finite nonempty set S of places of $\mathbb{Q}$,*
(a) (Weak approximation property) $\mathbb{Q}$ is dense in $\mathbb{Q}_S$ via the diagonal embedding, and
(b) (Strong approximation property) $\mathbb{Q}$ is dense in $\mathbb{A}^S$ via the diagonal embedding.

Proof. (a) Without loss of generality, we assume S contains the infinite place. We need to show that any open set in $\mathbb{Q}_S$ contains a nonzero element in $\mathbb{Q}$. Indeed, a basis of open sets of $\mathbb{Q}_S$ consists of open sets $U \times \prod_{p \in S, p < \infty} (a_p + p^{k_p} \mathbb{Z}_p)$ where U is open in $\mathbb{R}$ and $k_p \in \mathbb{Z}, a_p \in \mathbb{Q}$. We choose $x \in U \cap \mathbb{Q}$. Then by the Chinese Remainder Theorem, there exists $y \in \mathbb{Q}, z \in \mathbb{Z} \setminus \{0, 1\}$ such that $y \equiv a_p - x \pmod{p^{k_p}}$ and $z \equiv 1 \pmod{p^{k_p}}$ for all $p \in S \setminus \{\infty\}$. Hence, for sufficiently small $\ell < 0$, $x + yz^\ell$ is our desired element in $\mathbb{Q}$.

(b) We first consider the case where S does contain the infinite place. A basis of open sets of $\mathbb{A}^S$ consists of open sets $\prod_{p \in T} (a_p + p^{k_p} \mathbb{Z}_p) \times \prod_{p \notin S \cup T} \mathbb{Z}_p$ where T is a finite set of places of $\mathbb{Q}$, $T \cap S = \varnothing$, $a_p \in \mathbb{Q}$ for all $p \in T$. By the Chinese Remainder Theorem, there exists $x \in \mathbb{Q}$ such that $x \equiv a_p \pmod{p^{k_p}}$ where the denominator of x only has prime powers of primes $p \in T$. It follows x lies in the open set.

If S does not contain the infinite place, then there exists a prime $q \in S$. An open set of $\mathbb{A}^S$ consists of open sets $U \times \prod_{p \in T} (a_p + p^{k_p} \mathbb{Z}_p) \times \prod_{p \notin S \cup T} \mathbb{Z}_p$ where T is a finite set of places of $\mathbb{Q}$, $T \cap S = \varnothing$, $a_p \in \mathbb{Q}$ for all $p \in T$, U is open in $\mathbb{R}$.

There exists $\ell \in \mathbb{Z}_{>0}, x \in \mathbb{Z}$ such that $\frac{x}{q^\ell} \in U$. Indeed, pick any $y \in U$ and let ℓ be sufficiently large such that $(y - q^{-\ell}, y + q^{-\ell}) \subset U$. As $\left[q^\ell y - \frac{1}{2}, q^\ell y + \frac{1}{2}\right]$ has length 1, there exists an integer x lying inside that interval, giving $xq^{-\ell} \in U$.

By the Chinese Remainder theorem, there exists $z \in \mathbb{Q}, t \in \mathbb{Z}_{>1}$ such that $z \equiv a_p - xq^{-\ell} \pmod{p^{k_p}}$ and $q^t \equiv 1 \pmod{p^{k_p}}$ for all $p \in T$, where the denominator of z only has prime powers of primes $p \in T$. It follows $xq^{-\ell} + zq^{tk} \in \mathbb{Q}$ lies in the desired open set. □

Corollary 73. *Let $S = \{\infty\}$ be the set of all infinite places of $\mathbb{Q}$ then*

(a) $|\mathbb{Q} \setminus \mathbb{A}^{\infty} / \widehat{\mathbb{Z}}| = 1$

(b) We have an isomorphism $\mathbb{Q} \setminus \mathbb{A}/\widehat{\mathbb{Z}} \cong \mathbb{Z} \setminus \mathbb{R}$ *of topological spaces.*

Proof. (a) It suffices to prove that $\mathbb{A}^{\infty} = \mathbb{Q} + \widehat{\mathbb{Z}}$. Consider $x \in \mathbb{A}^{\infty}$. Then $x + \widehat{\mathbb{Z}}$ is an open neighborhood of x as $\widehat{\mathbb{Z}}$ is open subgroup in $\mathbb{A}^{\infty}$. From strong approximation theorem for adeles, we know $\mathbb{Q}$ is dense in $\mathbb{A}^{\infty}$. Hence, there exists $\ell \in \mathbb{Q}$ so that $\ell \in x + \widehat{\mathbb{Z}}$, implying $x \in \mathbb{Q} + \widehat{\mathbb{Z}}$.

(b) We identify $\mathbb{A}$ with $\mathbb{R} \times \mathbb{A}^{\infty}$. Consider the map

$$\begin{aligned}\phi : \mathbb{Z} \setminus \mathbb{R} &\to \mathbb{Q} \setminus \mathbb{A}/\widehat{\mathbb{Z}} \\ \mathbb{Z} + x &\mapsto [x, 0]\end{aligned}$$

where we denote $[x, y]$ for $(x, y) \in \mathbb{R} \times \mathbb{A}^{\infty}$ to be the double coset $\mathbb{Q} + (x, y) + \widehat{\mathbb{Z}}$. Note that $\widehat{\mathbb{Z}}$ and $\mathbb{Q}$ are embedded diagonally into $\mathbb{R}$ and $\mathbb{A}$, respectively.

We first show that ϕ is injective. If $x \in \mathbb{R}$ so $[x, 0] = 0$ then $(x, 0) = (\ell, \ell + y)$ for $y \in \widehat{\mathbb{Z}}$ and $\ell \in \mathbb{Q}$. It follows $\ell = -y \in \widehat{\mathbb{Z}}$ so $\ell \in \mathbb{Q} \cap \widehat{\mathbb{Z}} = \mathbb{Z}$. Hence, $x = \ell \in \mathbb{Z}$, as desired.

To show ϕ is surjective. From (a), we find $\mathbb{A}^{\infty} = \mathbb{Q} + \widehat{\mathbb{Z}}$. Hence, $\mathbb{A} = \mathbb{Q} + \mathbb{R} + \widehat{\mathbb{Z}}$ and surjectivity follows.

To show ϕ is continuous. Consider $U \subset \mathbb{R}$ to be representatives of an open subset U' in $\mathbb{Q} \setminus \mathbb{A}/\widehat{\mathbb{Z}}$ with $0 \in U$. Then $\mathbb{Q} + (U, \widehat{\mathbb{Z}})$ is open in $\mathbb{A}$. As $\mathbb{A} = \mathbb{R} \times \mathbb{A}^{\infty}$, we can cover $\mathbb{Q} + (U, \widehat{\mathbb{Z}})$ by open subsets (X_i, Y_i) where X_i and Y_i are open in $\mathbb{R}$ and $\mathbb{A}^{\infty}$, respectively. As $0 \in U$ and $\widehat{\mathbb{Z}}$ is open in $\mathbb{A}^{\infty}$, we find $\mathbb{Q} + (X_i, \widehat{\mathbb{Z}})$ is also open in $\mathbb{Q} + (U, \widehat{\mathbb{Z}})$ and these subsets cover $\mathbb{Q} + (U, \widehat{\mathbb{Z}})$.

On the other hand, as $\mathbb{Z} = \mathbb{Q} \cap \widehat{\mathbb{Z}}$, $\mathbb{Q} + (U, \widehat{\mathbb{Z}})$ is disjoint union of $\ell + (\mathbb{Z} + U, \widehat{\mathbb{Z}})$ for $\ell \in \mathbb{Q}, \ell \notin \mathbb{Z}$. Combining with the previous argument, we find $(\mathbb{Z} + U, \widehat{\mathbb{Z}})$ must be obtained from taking the union of $(X_i, \widehat{\mathbb{Z}})$, where X_i are open subsets of $\mathbb{R}$. It follows $U + \mathbb{Z}$ is open in $\mathbb{R}$, meaning inverse image of U' under ϕ is open in $\mathbb{Z} \setminus \mathbb{R}$.

To show ϕ has continuous inverse, it suffices to show ϕ is open map. Consider $U \subset \mathbb{R}$ so $\mathbb{Z} + U$ is open in $\mathbb{R}$ and we need to show $\mathbb{Q} + U + \widehat{\mathbb{Z}}$ is open in $\mathbb{A}$. This holds because $\mathbb{Q} + U + \widehat{\mathbb{Z}}$ is the union of open sets $\ell + (\mathbb{Z} + U, \widehat{\mathbb{Z}})$ where $\ell \in \mathbb{Q}$. □

Corollary 74. *The quotient* $\mathbb{Q} \setminus \mathbb{A}$ *is compact.*

Proof. From the previous corollary, we obtain a homeomorphism of topological spaces

$$\mathbb{Q} \setminus \mathbb{A} \cong \mathbb{Z} \setminus \mathbb{R} \times \widehat{\mathbb{Z}}.$$

Note that $\widehat{\mathbb{Z}}$ is compact. As $\mathbb{Z}$ is a lattice in $\mathbb{R}$, $\mathbb{Z} \setminus \mathbb{R}$ is compact. Thus, $\mathbb{Q} \setminus \mathbb{A}$ is compact. □

5.2. **Topology of adelic points.** Let X be an affine k-scheme of finite type. For a k-algebra R which is also a topological ring, we can endow $X(R)$ with a canonical topology. When $k = \mathbb{Q}$ and $R = \mathbb{A}$, $X(\mathbb{A})$ is homeomorphic to the restricted product of $X(\mathbb{Q}_v)$ over all places v of $\mathbb{Q}$.

Proposition 75. *Let R be a topological ring. There exists a unique way to topologise $X(R)$ for all affine schemes X of finite type over R such that*

(1) the topology is functorial in X; that is, if $X \to Y$ is a morphism of affine schemes of finite type of R, then the induced map on points $X(R) \to Y(R)$ is continuous;

(2) the topology is compatible with fiber products: this means if $X \to Y$ and $Y \to Z$ are morphisms of affine schemes of finite type over R, then the topology on $(X \times_Z Y)(R)$ is the fiber product topology;

(3) closed immersion of affine schemes $X \hookrightarrow Y$ (i.e. the map of coordinate rings $\mathcal{O}(Y) \to \mathcal{O}(X)$ is surjective) induces topological embeddings $X(R) \hookrightarrow Y(R)$ (i.e. a continuous map that is homeomorphic onto its image);

(4) if $X = \operatorname{Spec} R[t]$ then $X(R)$ is homeomorphic with R under the natural identification $X(R) \cong R$.

If R is Hausdorff or locally compact, then so is $X(R)$. Moreover, if R is Hausdorff then the closed immersion $X \to Y$ induces a closed embedding $X(R) \to Y(R)$.

Sketch of proof. We refer to [Con12] for the proofs of these two propositions. Essentially, the topology of $X(R)$ is constructed by choosing an R-algebra isomorphism $\mathcal{O}(X) \cong R[t_1, \dots, t_n]/I$ for the coordinate ring of X, for any ideal I. The set $X(R)$ can then be identified with the sets of elements in R^n on which the elements in I (we view elements of I as R-valued functions on R^n) all vanish. We have an injection $X(R) \hookrightarrow R^n$ which we equip $X(R)$ with the subspace topology of R^n. One then has to check that the defined topology does not depend on the choice of isomorphism $\mathcal{O}(X) \cong R[t_1, \dots, t_n]/I$ and satisfies all the functorial properties as above. ☐

Proposition 76. *Let $R \to R'$ be a continuous map of topological rings and let X be an affine scheme of finite type over R. Then $X(R) \to X(R')$ is continuous. Moreover, if $R \to R'$ is a*

(1) a topological embedding,
(2) a open topological embedding,
(3) a closed topological embedding,
(4) a topological embedding onto a discrete subset,

then so is $X(R) \to X(R')$.

Proof. The proposition follows from the commutative diagram

$$\begin{array}{ccc} X(R) & \overset{i}{\hookrightarrow} & R^n \\ \downarrow f & & \downarrow g \\ X(R') & \overset{i'}{\hookrightarrow} & R'^n \end{array}$$

where i and i' are topological embeddings. For example, we find $f^{-1}(U) = X(R) \cap g^{-1}(U)$ for any $U \subset X(R')$ so f is continuous. If g is a topological embedding then so is $g \circ i$, hence f is also a topological embedding. If g is open or closed then so is f. ☐

Example 77 (Topology of adeles and ideles). When $k = \mathbb{Q}$ and $R = \mathbb{A}$, we have $\mathbb{G}_a(R) = \mathbb{A}$. From § 5, we know that $\mathbb{A}$ has the topology generated by the basis of open sets

$$U_S \times \prod_{v \notin S} \mathbb{Z}_v$$

where S is a finite set of places of k and U_S is open in $\mathbb{Q}_S = \prod_{v \in S} \mathbb{Q}_v$.

Next, we consider the ideles $\mathbb{G}_m(\mathbb{A}) = \mathbb{A}^\times$. We have a closed immersion $\mathbb{G}_m \hookrightarrow \mathbb{G}_a \times \mathbb{G}_a$ sending $t \mapsto (t, t^{-1})$. Therefore, we have a topological embedding $\mathbb{A}^\times \hookrightarrow \mathbb{A} \times \mathbb{A}$, giving $\mathbb{G}_m(\mathbb{A}) = \mathbb{A}^\times$ the topology generated by the basis of open sets

$$U_S \times \prod_{v \notin S} \mathbb{Z}_v^\times$$

where S is a finite set of places of k and U_S is open in $\mathbb{Q}_S^\times = \prod_{v \in S} \mathbb{Q}_v^\times$ under the product topology [7]. Note that this is not the same topology as giving $\mathbb{A}^\times \subset \mathbb{A}$ the subspace topology. In particular, $\prod_{p<\infty} \mathbb{Z}_p^\times$ is an open set in $\mathbb{A}^\times$ but it is not open under the subspace topology from $\mathbb{A}$. Indeed, if $\prod_{p<\infty} \mathbb{Z}_p^\times$ is open under the subspace topology from $\mathbb{A}$, it is a union of $\mathbb{A}^\times \cap U$'s where $U = U_S \times \prod_{v \notin S} \mathbb{Z}_v$ is open in $\mathbb{A}$. One can then choose a sufficiently large prime p such that $a = (a_v)_v$

[7] To see this, consider open set $U \times V$ in $\mathbb{A} \times \mathbb{A}$. For $a = (a_v) \in \mathbb{A}$, if $(a, a^{-1}) \in U \times V$ then $a_v^{\pm 1} \in \mathbb{Z}_v$ for almost all v, meaning $a_v \in \mathbb{Z}_v^\times$ for almost all v; finally, note that $\mathbb{Q}_v^\times$ has the subspace topology from $\mathbb{Q}_v$ for all places v

satisfies $(a_v)_{v\in S} \in U_S \cap \mathbb{Q}_S^\times$, $a_p = p$ and $a_v = 1$ for $v \notin S \cup \{p\}$. It follows $a \notin \prod_{v<\infty} \mathbb{Z}_p^\times$ but $a \in \mathbb{A}^\times \cap U$, a contradiction.

Thus, we have a homeomorphism

$$\mathbb{G}_m(\mathbb{A}) \cong \prod_v{}' \mathbb{G}_m(\mathbb{Q}_v).$$

Example 78 (Topology of $\mathrm{GL}_2(\mathbb{A})$). The map $\mathrm{GL}_2 \hookrightarrow M_2 \times \mathbb{G}_a$ sending $x \mapsto (x, \det^{-1} x)$ is a closed immersion of affine schemes since the associated k-algebra map $k[x_{11}, x_{12}, x_{21}, x_{22}] \otimes_k k[t] \to k[x_{11}, x_{12}, x_{21}, x_{22}, \det^{-1}]$ sending t to $\det^{-1} := (x_{11}x_{22} - x_{12}x_{21})^{-1}$ is surjective. Hence, we have a topological embedding $\mathrm{GL}_2(R) \hookrightarrow M_2(R) \times \mathbb{G}_a(R)$.

With the above embedding, we will describe the topology of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. It suffices to describe a basis of open neighborhoods of the identity of $\mathrm{GL}_2(\mathbb{A})$.

We first describe the topology of $\mathrm{GL}_2(\mathbb{A}_T)$ where $\mathbb{A}_T := \mathbb{Q}_T \times \widehat{\mathbb{Z}}^T = \prod_{v\in T} \mathbb{Q}_v \times \prod_{v\notin T} \mathbb{Z}_v$ for some fixed finite set T of places of $\mathbb{Q}$ containing the infinite place. We know that $(I_2 + p^k M_n(\mathbb{Z}_p)) \times (1 + p^k\mathbb{Z}_p)$ for $k \in \mathbb{Z}_{\geq 1}$ forms a basis of open neighborhoods of $(I_2, 1) \in M_2(\mathbb{Q}_p) \times \mathbb{Q}_p$. Therefore, for any finite set S of places of $\mathbb{Q}$ containing the infinite place, and $k \in \mathbb{Z}_{\geq 1}$, the collection

$$\prod_{p\in S\cap T} \Big((I_2 + p^k M_2(\mathbb{Z}_p)) \times (1 + p^k\mathbb{Z}_p)\Big) \times \prod_{p\notin S\cup T} (M_2(\mathbb{Z}_p) \times \mathbb{Z}_p)$$

forms a basis of open neighborhoods of $(I_2, 1)$ in $M_2(\mathbb{A}_T) \times \mathbb{A}_T$ (by definition of the product topology). Intersecting these sets with the image of $\mathrm{GL}_2(\mathbb{A}_T)$ from the embedding, we obtain

$$\prod_{p\in S\cap T} (I_2 + p^k M_2(\mathbb{Z}_p)) \times \prod_{p\notin S\cup T} \mathrm{GL}_2(\mathbb{Z}_p)$$

as a basis of open neighborhoods of I_2 in $\mathrm{GL}_2(\mathbb{A})$. Thus, we have a homeomorphism

$$\mathrm{GL}_2(\mathbb{A}_T) \cong \prod_{v\in T} \mathrm{GL}_2(\mathbb{Q}_v) \times \prod_{v\notin T} \mathrm{GL}_2(\mathbb{Z}_v).$$

Now, $\mathbb{A}_T \hookrightarrow \mathbb{A}$ is an open embedding, so $\mathrm{GL}_2(\mathbb{A}_T) \hookrightarrow \mathrm{GL}_2(\mathbb{A})$ is also an open topological embedding. Furthermore, as $\mathrm{GL}_2(\mathbb{A}) = \bigcup_S \mathrm{GL}_2(\mathbb{A}_S)$ over all finite set S of places of $\mathbb{Q}$ containing the infinite place, we conclude that the collection

$$\prod_{v\in S} U_v \times \prod_{v\notin S} \mathrm{GL}_2(\mathbb{Z}_v)$$

forms a basis of open sets of $\mathrm{GL}_2(\mathbb{A})$, where S is a finite set of places of $\mathbb{Q}$ containing the infinite place and U_v is open in $\mathrm{GL}_2(\mathbb{Q}_v)$.Thus, we have a homeomorphism

$$\mathrm{GL}_2(\mathbb{A}) \cong \prod_v{}' \mathrm{GL}_2(\mathbb{Q}_v).$$

The argument in the previous example holds for general GL_n, giving a homeomorphism $\mathrm{GL}_n(\mathbb{A}) \cong \prod_v' \mathrm{GL}_n(\mathbb{Q}_v)$. Hence, we have the following result.

Proposition 79. *For a linear algebraic group G over $\mathbb{Q}$ and a faithful representation $G \to \mathrm{GL}_n$, one has an isomorphism of topological groups*

$$G(\mathbb{A}) \cong \prod_v{}' G(\mathbb{Q}_v)$$

where the restricted product on the right is defined with respect to the compact open subgroup $G(\mathbb{Q}_v) \cap \mathrm{GL}_n(\mathbb{Z}_v)$ of $G(\mathbb{Q}_v)$, where $v \nmid \infty$.

Example 80 (Topology of $\mathrm{SL}_2(\mathbb{A})$). We have a closed immersion $\mathrm{SL}_2 \hookrightarrow \mathrm{GL}_2$ of affine schemes as $\mathcal{O}(\mathrm{SL}_2) = \mathcal{O}(\mathrm{GL}_2)/(x_{11}x_{22} - x_{12}x_{11} - 1)$, giving a topological embedding $\mathrm{SL}_2(R) \hookrightarrow \mathrm{GL}_2(R)$ for any topological ring R.

Since $\mathbb{Q}$ is a discrete subgroup of $\mathbb{A}_{\mathbb{Q}}$, $\mathrm{SL}_2(\mathbb{Q})$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{A})$.

We refer to [Con12] for more discussions on describing the adelic topology of $X(\mathbb{A}_k)$ for any separated scheme X of finite type over k. For example, on [Con12, p. 10], when removing the affineness assumption of X, $X(k)$ may not be a discrete closed subset of $X(\mathbb{A}_k)$.

5.3. **Analytic manifolds from smooth schemes.** Continuing from Proposition 75, we restrict X to be a smooth affine scheme of finite type over k, where k is a complete valued field. In this section, we show that one can endow $X(k)$ with a canonical structure of a k-analytic manifold. Here smoothness of X means the following (see [GW20, §6.8])

Definition 81. *Let X be an affine scheme of finite type over k. We say X is* smooth of dimension d over k *if X can be covered by affine open sets $\operatorname{Spec} k[t_1, \dots, t_n]/(f_1, \dots, f_{n-d})$ for suitable n and f_i, such that the Jacobian matrix $(\partial f_i/\partial t_j)(x) \in M_{(n-d)\times n}(\kappa(x))$ has rank $n-d$. Equivalently, the ideal in $k[t_1, \dots, t_n]$ generated by the f_i's and all the $(n-d)\times(n-d)$ minors of the Jacobian $(\partial f_i/\partial t_j)$ is the whole ring $k[t_1, \dots, t_n]$.*

Example 82. $\mathrm{GL}_2 = \operatorname{Spec} k[x_{11}, x_{12}, x_{21}, x_{22}, t]/(t(x_{11}x_{22} - x_{21}x_{12}) - 1)$ is a smooth scheme over k of dimension 4.

Example 83. $\mathrm{SL}_2 = \operatorname{Spec} k[x_{11}, x_{12}, x_{21}, x_{22}]/(x_{11}x_{22} - x_{21}x_{12} - 1)$ is a smooth scheme over k of dimension 3.

Proposition 84. *There is a canonical structure of a k-analytic manifold on $X(k)$, which is characterised as follows:*

1. *Functorial in X: morphism of smooth k-schemes induce morphism of k-analytic manifolds; open (resp. closed) immersions induce open (resp. closed) immersions of k-analytic manifolds.*
2. *When $X = \operatorname{Spec} k[x_1, \dots, x_d]$, the structure of k-analytic manifold on $X(k) \cong k^d$ is the natural one.*
3. *Etale morphisms of smooth k-schemes induce k-analytic local isomorphisms.*

Sketch of proof. Defining a structure of a k-analytic manifold on $X(k)$ amounts to describing which continuous functions are k-analytic, such that the induced locally ringed space is locally isomorphic to k^n with its sheaf of k-analytic functions. Let U be an open set of $X(k)$. A continuous function $f : U \to k$ is k-analytic at $x \in U$ if there exists an immersion of k-schemes $i : V \to \operatorname{Spec} k[t_1, \dots, t_n]$ on a Zariski-open neighborhood V of x in X and a k-analytic function $g : W \to k$ on an open neighborhood of x such that $f = g \circ i$ on some open neighborhood of x in U. We say f is analytic if it is analytic at every point in U.

We refer to [CLNS18, Chapter 0, §1.6] for the verification of the functorial conditions with the above analytic structure. □

Example 85. If $k = \mathbb{Q}_p$ or $k = \mathbb{R}$ then $\mathrm{GL}_2(k), \mathrm{SL}_2(k)$ are k-analytic manifolds.

5.4. **Weil measure on the integral points of nonarchimedean local fields.** From the previous section, for a smooth scheme X of finite type over a nonarchimedean local field k, $X(k)$ is a k-analytic manifold. In this section, we show that if X a smooth scheme over $\mathcal{O}_k$, there is a very natural measure on $X(\mathcal{O}_k)$, called the Weil measure, that does not depend on the choice of a volume form, i.e. a nowhere vanishing differential form of top degree. Furthermore, if we can construct *any* measure on $X(k)$ by integrating a volume form on $X(k)$ (see § 4.4), the Weil measure is then the restriction measured on $X(\mathcal{O}_k) \subset X(k)$. In this section, we will also prove a theorem of Weil that links point counting over finite fields with integration over local fields. We will mainly follow two references [CLNS18, Chapter 0] and [Mag16].

Let k be a nonarchimedean local field, $\mathcal{O}_k$ be its ring of integers, $\mathbb{F}_q$ be the residue field where q is a prime power of a prime p. Let X be a smooth scheme of relative dimension n over $\mathcal{O}_k$ and $\Omega_{X/\mathcal{O}_k}$ be the sheaf of differentials.

We will define a canonical measure on $X(\mathcal{O}_k)$, called the *Weil measure*. Because X is smooth over $\mathcal{O}_k$ and $\Omega^n_{X/\mathcal{O}_k}$ is a locally free sheaf of $\mathcal{O}_X$-modules of rank 1, there exists an affine open cover $\{U_i\}$ of $\mathcal{O}_k$-schemes of X such that we have a trivialisation $\Omega^n_{X/\mathcal{O}_k}|_{U_i} \cong \mathcal{O}_X|_{U_i}$ over each U_i. A trivialisation of $\Omega^n_{X/\mathcal{O}_k}|_{U_i}$ corresponds to a nowhere-vanishing differential form $\omega_i \in \Gamma(U_i, \Omega^n_{X/\mathcal{O}_k})$. From this, we can define a (Radon) measure $d|\omega_i|$ on $U_i(\mathcal{O}_k)$ by integrating with respect to ω_i. We also have $X(\mathcal{O}_k) = \bigcup_i U_i(\mathcal{O}_k)$, so in order to define a (Radon) measure on $X(\mathcal{O}_k)$, the measures $d|\omega_i|$ must agree on overlaps. This is true because for two nowhere-vanishing differential forms $\omega_i|_{U_i\cap U_j}$ and $\omega_j|_{U_i\cap U_j}$ on $U_i \cap U_j$, there exists a nowhere-vanishing function $f \in \mathcal{O}_X|_{U_i\cap U_j}$ (hence invertible) so that $\omega_i = f\omega_j$ on $U_i \cap U_j$. This gives us the relation $d|\omega_i|(x) = |f(x)|_k d|\omega_j|(x)$ of measures on $U_j(\mathcal{O}_k) \cap U_i(\mathcal{O}_k) = (U_i \cap U_j)(\mathcal{O}_k)$. However, as $f : (U_i \cap U_j)(\mathcal{O}_k) \to \mathcal{O}_k$ is invertible, $|f(x)|_k = 1$ for all $x \in (U_i \cap U_j)(\mathcal{O}_k)$, meaning $d|\omega_i| = d|\omega_j|$ on $(U_i \cap U_j)(\mathcal{O}_k)$.

Remark 86. The Weil measure is canonical in the sense that its construction does not depend on the existence of a global differential form. The main reason for this is that our scheme X is over $\mathcal{O}_k$, hence any invertible function f, defined on an open set U of X, must have $|f(x)|_k = 1$ for all $x \in U(\mathcal{O}_k)$. This also means that one may not be able to repeat this construction to define a measure on $X(k)$. However, if we have a global differential form $\omega \in \Gamma(X, \Omega^n_{X/\mathcal{O}_k})$, we can define a measure on $X(k)$ whose restriction to $X(\mathcal{O}_k)$ is the Weil measure.

In the literature (see [Bat99]), it seems that the name Weil measure is given when X has a global nowhere-vanishing differential form, and the measure we have constructed is called the canonical measure. In fact, the two measures are the same if the Weil measure (as defined in the literature) exists. Thus, for convenience, we will stick with our definition of Weil measure.

Theorem 87 (Weil)**.** *Let X be a smooth scheme of dimension n over $\mathcal{O}_k$. Let μ be the Weil measure on $X(\mathcal{O}_k)$, then*

$$\int_{X(\mathcal{O}_k)} d\mu = \frac{|X(\mathbb{F}_q)|}{q^n}.$$

Sketch. Because X is smooth over $\mathcal{O}_k$, the reduction map $\varphi : X(\mathcal{O}_k) \to X(\mathbb{F}_q)$ sending $x \mapsto \overline{x}$ is surjective, giving

$$\int_{X(\mathcal{O}_k)} d\mu = \sum_{\overline{x}\in X(\mathbb{F}_q)} \int_{\varphi^{-1}(\overline{x})} d\mu.$$

It suffices to show $\int_{\varphi^{-1}(\overline{x})} d\mu = q^{-n}$ for all $\overline{x} \in X(\mathbb{F}_q)$. We view $\overline{x} \in X(\mathbb{F}_q)$ as an element of X by taking the value $\overline{x}(\eta)$ at the generic point $\eta \in \operatorname{Spec} \mathbb{F}_q$. Because X is smooth and $\Omega_{X/\mathcal{O}_k}$ is locally free, there exists an affine open set $U \cong \operatorname{Spec} \mathcal{O}_k[x_1, \dots, x_{n+m}]/(f_1, \dots, f_m)$ of $\overline{x}$ such that $\Omega_{X/\mathcal{O}_k}|_U$ is trivialised and the Jacobian matrix $(\partial f_i/\partial x_{n+j})_{1\le i,j\le m}$ is invertible at $\varphi^{-1}(\overline{x}) \subset U(\mathcal{O}_k)$ ($\varphi^{-1}(\overline{x}) \subset U$ as any open set of $\overline{x}$ contains $x \in \varphi^{-1}(\overline{x})$, viewed an element of X by evaluating at the generic point $\eta \in \operatorname{Spec} \mathcal{O}_k$). We consider the map $g : U(\mathcal{O}_k) \to \mathbb{A}^{n+m}_{\mathcal{O}_k}$ defined by $g(x_1, \dots, x_{n+m}) = (x_1, \dots, x_n, f_1(x), \dots, f_m(x))$. Observe that the Jacobian of g at $\varphi^{-1}(\overline{x})$ is a unit in $\mathcal{O}_k$. Therefore, by forgetting the last m coordinates, g induces an etale morphism $h : U \to \mathbb{A}^n_{\mathcal{O}_k}$, which induces a k-analytic isomorphism from $\varphi^{-1}(\overline{x})$ to $\mathfrak{p}^n$ by Hensel's lemma, where $\mathfrak{p}$ is the maximal ideal of $\mathcal{O}_k$. Furthermore, because $\Omega_{X/\mathcal{O}_k}|_U \cong \mathcal{O}^n_X|_U$, we can find a global nowhere-vanishing differential form $\omega \in \Gamma\left(U, \bigwedge^n \Omega_{X/\mathcal{O}_k}|_U\right)$. We then have $h^*(dt_1 \wedge dt_2 \wedge \cdots \wedge dt_n) = f\omega$, where f is invertible in U, hence

$f(x)$ p-adic norm for $x \in U(\mathcal{O}_k)$. By definition, $f\omega$ defines a Weil measure on the neighborhood $U(\mathcal{O}_k)$. Thus, we find

$$\int_{\varphi^{-1}(\overline{x})} d\mu = \int_{\mathfrak{p}^n} dt_1 \wedge \cdots dt_n = q^{-n}$$

by the change of variables formula. □

Proposition 88. *Let X be a smooth scheme over $\mathcal{O}_k$, Y be a reduced closed subscheme of codimension at least 1 in X. Then $Y(\mathcal{O}_k)$ has measure 0 in $X(\mathcal{O}_k)$ with respect to the Weil measure.*

Sketch. By using affine cover of X, we can reduce the problem to the case where X is an affine smooth scheme. By considering some hypersurface containing Y, we can also reduce to the case of principal divisor, i.e. Y is defined by $f = 0$ for irreducible $f \in \Gamma(X, \mathcal{O}_X)$. By the Noether normalisation theorem, we can then further assume that $X = \operatorname{Spec} \mathcal{O}_k[x_1, \ldots, x_n]$ and $f = x_1$.

To show $\mu(Y(\mathcal{O}_k))$ where μ is the Weil measure, for $m \in \mathbb{Z}_{\geq 1}$, we set

$$Y_m(\mathcal{O}_k) := \{(x_1, \ldots, x_n) \in \mathcal{O}_k^n : x_1 \in \mathfrak{m}^m\}.$$

We then find

$$Y(\mathcal{O}_k) = \bigcap_{m=1}^{\infty} Y_m(\mathcal{O}_k).$$

We also have

$$\mu(Y_m(\mathcal{O}_k)) = \int_{\mathfrak{m}^m} |dx_1| \prod_{i=2}^{m} \int_{\mathcal{O}_k} |dx_i| = q^{-m}.$$

Therefore, we find

$$\mu(Y(\mathcal{O}_k)) = \lim_{m\to\infty} \mu(Y_m(\mathcal{O}_k)) = 0.$$

□

5.5. **Measure on the adelic points.** Let k be a global field, X be a smooth affine scheme of relative dimension n over k, $\omega \in \Gamma(X, \Omega^n_{X/\operatorname{Spec} k})$ be a global nowhere vanishing algebraic differential form over k, called a *volume form*. Our goal in this section is to associate a measure on $X(\mathbb{A}_k)$.

5.5.1. *Measures on local points.* We first normalise the Haar measures on each local field k_v as follows:

(1) If $k_v = \mathbb{R}$ then we use the Lebesgue measure.
(2) If $k_v = \mathbb{C}$ then we use twice the standard Lebesgue measure on $\mathbb{C} \cong \mathbb{R}^2$, i.e. if $z = x + iy$ with $x, y \in \mathbb{R}$ then this Haar measure is $|dz \wedge d\overline{z}| = 2dx \wedge dy$ where dx and dy are the Lebesgue measure on $\mathbb{R}$.
(3) If k_v is nonarchimedean, we normalise the Haar measure on k_v so that $\mathcal{O}_v$ has volume 1.

By arguing similarly as in § 4.4 and § 5.4, for each place v of k, we can construct a (Radon) measure $|\omega|_v$ on $X(k_v)$ as follows:

(1) Cover X by affine schemes U_i's with chart $(U_i, \mathcal{O}_X|_{U_i}) \xrightarrow{\sim} (V \subset k_v^n, \mathcal{O}_{k_v^n}|_V)$. When taking k_v-points, there holds $X(k_v) = \bigcup_i U_i(k_v)$. On $U_i(k_v)$, with local coordinates $x_1, \ldots, x_n$ coming from the isomorphism $(U_i, \mathcal{O}_X|_{U_i}) \xrightarrow{\sim} (V \subset k_v^n, \mathcal{O}_{k_v^n}|_V)$, ω can then be written as $f dx_1 \wedge \cdots \wedge dx_n$ where f is k-analytic on $U_i(k_v)$.
(2) We can define a measure $|f|_v dx_1 dx_2 \cdots dx_n$ on each $U(k_v)$. To define a measure on $X(k_v)$, we use partition of unity. One then shows that the resulting measure $|\omega|_v$ on $X(k_v)$ does not depend on the choice of local coordinates or on the choice of partitions of unity.

Example 89. For SL_2 over $\mathbb{Q}$, as we did in the proof of Proposition 112, we can choose $U_{11} = \operatorname{Spec} \mathcal{O}(\mathrm{SL}_2)_{x_{11}}$ and $U_{12} = \operatorname{Spec} \mathcal{O}(\mathrm{SL}_2)_{x_{12}}$ to cover $\operatorname{Spec} \mathcal{O}(\mathrm{SL}_2)$. Hence, ω over $U_{11}(\mathbb{Q}_v) = \{(x_{11}, x_{12}, x_{21}) \in \mathbb{Q}_v^3 : x_{11} \neq 0\}$ can be written as $\frac{1}{x_{11}} dx_{11} \wedge dx_{12} \wedge dx_{21}$, and indeed x_{11}^{-1} is $\mathbb{Q}_v$-analytic on $U_{11}(\mathbb{Q}_v)$.

5.5.2. *Local finiteness of the measure on $X(\mathbb{A}_k)$.* We would like to construct a Radon measure on our locally compact space $X(\mathbb{A}_k)$ roughly as a product of measures $|\omega|_v$ on $X(k_v)$. In particular, we require our measure to have finite volume on compact sets. In order to address this issue, we will describe a family of compact sets of $X(\mathbb{A}_k)$ via taking certain integral model of X, and then use this information to give criteria that ensures the existence of a Radon measure on $X(\mathbb{A})$.

For each place v of k, let k_v be the completion of k with respect to v and $\mathcal{O}_v$ be its ring of integers and $\kappa(v)$ be the corresponding residue field. Let S be a nonempty finite set of places of k containing all archimedean places and let $\mathcal{O}_S := \{x \in k : x \in \mathcal{O}_v \forall v \notin S\}$. For example, if $k = \mathbb{Q}, S = \{2, 3\}$ then $\mathcal{O}_S = \mathbb{Z}[\frac{1}{6}]$.

By the principle of "spreading out" (see [Poo17, §3.2]), by enlarging S if necessary, there exists a smooth scheme $\overline{X}$ of finite type over $\mathcal{O}_S$ such that $\overline{X} \times_{\operatorname{Spec} \mathcal{O}_S} \operatorname{Spec} k = X$ [8] and ω extends to a volume form $\overline{\omega}$ on $\overline{X}$. As $\operatorname{Spec} \mathcal{O}_v \to \operatorname{Spec} \mathcal{O}_S$ for $v \notin S$, we can regard $\overline{X}(\mathcal{O}_v)$ as a compact open subset of $X(k_v) = \overline{X}(k_v)$, and

$$\prod_{v \notin S'} \overline{X}(\mathcal{O}_v) \times \prod_{v \in S'} X(k_v) \tag{2}$$

as an open subset of $X(\mathbb{A}_k)$ for any finite set S' of places of k containing S.

Furthermore, from Theorem 87, we know $|\omega|_v(\overline{X}(\mathcal{O}_v)) = |\overline{\omega}|_v(\overline{X}(\mathcal{O}_v)) = \frac{|\overline{X}(\kappa(v))|}{|\kappa(v)|^n}$.

As every point in $X(\mathbb{A}_k)$ lies in the open subset (2), if the infinite product

$$\prod_{v \notin S} \frac{|\overline{X}(\kappa(v))|}{|\kappa(v)|^n}$$

converges absolutely, we can define a Radon measure $\mu_X = \prod_v{}' |\omega|_v$ on $X(\mathbb{A}_k)$ whose restriction to (2) is the product measure. If X satisfies this condition, we say X *admits an adelic measure*.

Note that this definition of μ_X (if it exists) does not depend on the chosen subset S of places of k, or on the choice of integral model $\overline{X}$ (because two choices of integral models become isomorphic after enlarging S).

5.5.3. *Weil restriction of scalars.* If X admits a Tamagawa measure, we normalise the measure on $X(\mathbb{A})$ to

$$\mu_X = \rho_k^{-\dim X} \prod_v{}' |\omega|_v,$$

where

1. if k is a number field, $\rho_k = |\Delta_k|^{1/2}$ where Δ_k is the discriminant of the field k, and
2. if k is the function field of a curve X over $\mathbb{F}_q$, $\rho_k = q^{g-1}$ where f is the genus of X.

There are two main reasons why we do this. The first reason is that in some sense, this normalisation makes the measure depend only on the scheme X rather than the choice of global field k. More precisely, we have the following

Proposition 90. *Let k' be a finite separable extension of a global field k, let X' be a smooth scheme of finite type over k' that admits an adelic measure, then its Weil restriction of scalars $X = Res_k^{k'} X'$, i.e. a scheme over k defined by $X(R) := X'(R \otimes_k k')$ for any k-algebra R, is a smooth scheme of finite type over k which also admits an adelic measure. Furthermore, we have a canonical homeomorphism*

$$X(\mathbb{A}_k) \cong X'(\mathbb{A}_{k'})$$

[8] A rough sketch of this idea over $\mathbb{Q}$: We first consider smooth affine scheme and choose equations for this scheme. Then there exists large enough n so that the coefficients are in $\mathbb{Z}[\frac{1}{n}]$. This gives us an integral model over $\mathbb{Z}[\frac{1}{n}]$. Next, consider any smooth scheme by gluing along affine opens. One needs to check smoothness along the way.

that is a measure-preserving morphism if we equip both sides with the normalised measures as discussed before.

For the proof of this proposition, we refer to [Wei82, p. 22]. We delay the second reason for this choice of normalisation to the section § 8.1 (after introducing Fourier analysis on adeles).

6. Fourier analysis on locally compact abelian groups

In this section, we study Fourier analysis on locally compact abelian groups, in particular for the cases of $\mathbb{R}, \mathbb{Q}_p$ and $\mathbb{A}$. Our goal is to establish the Poisson summation formula. The references we use for this section are [Fol16, §4] and [Poo15].

Throughout the section, let G be always a locally compact Hausdorff abelian topological group. For example, $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is a locally compact Hausdorff abelian group.

6.1. Pontryagin dual.

In this subsection, we will define the Pontryagin dual $\widehat{G}$ of a locally compact abelian group G. The Pontryagin duality then claims that G is isomorphic to $\widehat{\widehat{G}}$ as topological groups. Furthermore, when G is a local field or G is the adeles, we also have an isomorphism $G \cong \widehat{G}$ of topological groups.

Definition 91. *A* character *of G is a continuous homomorphism $\chi : G \to \mathbb{C}^\times$. A* unitary character *of G is a continuous homomorphism $\chi : G \to S^1$. The* Pontryagin dual *$\widehat{G}$ of G is the group of unitary characters of G, with the group operation being pointwise multiplication. We can equip $\widehat{G}$ with the* compact-open topology*, i.e. the topology generated by $\{\chi \in \widehat{G} : \chi(K) \subset U\}$ for every compact $K \subset G$ and open $U \subset S^1$.*

In fact, $\widehat{G}$ is also a locally compact abelian group. Any continuous homomorphism $\phi : G \to H$ of locally compact abelian groups induces a continuous homomorphism $\widehat{H} \to \widehat{G}$ taking χ to $\chi \circ \phi$. In fact, taking the Pontryagin dual is a contravariant and exact functor from the category of locally compact abelian groups to itself.

Example 92. If G is discrete then $\widehat{G}$ is compact. Indeed, the compact-open topology on $\widehat{G}$ is precisely the topology of pointwise convergence of all maps from G to S^1. With respect to the latter topology, $\widehat{G}$ is a closed subset of the space of all maps from G to S^1. The latter space is compact as it is homeomorphic to $(S^1)^{|G|}$, therefore $\widehat{G}$ is also compact.

We will assume the following result (see [Fol16, p. 110] for the proof):

Theorem 93 (Pontryagin duality)**.** *We have a canonical isomorphism of topological groups*

$$\begin{aligned} G &\to \widehat{\widehat{G}}, \\ g &\mapsto (\chi \mapsto \chi(g)). \end{aligned}$$

In the next subsections, we will explain the following table:

G	$\widehat{G}$
$\mathbb{R}$	$\mathbb{R}$
$\mathbb{Q}_p$	$\mathbb{Q}_p$
$\mathbb{A}$	$\mathbb{A}$
$\mathbb{Z}$	$\mathbb{R}/\mathbb{Z}$
$\mathbb{Z}_p$	$\mathbb{Q}_p/\mathbb{Z}_p$
$\mathbb{Q}$	$\mathbb{Q} \setminus \mathbb{A}$

6.1.1. *Pontryagin duals of local fields.* Let k be a local field.

Proposition 94. *For a local field k and a nontrivial unitary character ψ of $(k, +)$, we have an isomorphism $\Psi : k \to \widehat{k}$ of locally compact abelian groups, sending $a \mapsto \psi_a$, where $\psi_a(x) := \psi(ax)$.*

Proof. We check Ψ is injective. If $\psi_a = \psi_b$ for $a, b \in k$ then $\psi(ax) = \psi(bx)$ for all $x \in k$, or $\psi((a-b)x) = 1$ for all $x \in k$. As ψ is nontrivial, we find $a = b$.

We show that Ψ is a homeomorphism onto its image. From the topology of $\widehat{k}$, it suffices to show that k has $C(K,U) = \{a \in k : \psi_a(K) \subset U\} = \{a \in k : aK \subset \psi^{-1}(U)\}$ as basis of open neighborhoods of 0, where $K \subset k$ is compact and $1 \in U \subset S^1$ is open.

For any compact set K of k and open set U of S^1 containing 1, as $\psi^{-1}(U)$ contains an open disk around 0 and K is bounded, there exists $\delta > 0$ such that if $a \in k, |a| < \delta$ then $aK \subset \psi^{-1}(U)$. This shows $C(K,U)$ is open in k, as given $a_0 \in C(K,U)$, we know for all $a \in k$ such that $|a - a_0| < \delta$ then $(a - a_0)K \subset \psi^{-1}(U)$, implying $a \in C(K,U)$.

For any $\delta > 0$, we show that there exists a compact K of k and an open set U of S^1 containing 1 such that the open disk $|a| < \delta$ contains $C(K,U)$. Indeed, we can choose $b \in k$ such that $\psi(b) \neq 1$ (ψ is nontrivial) and choose open $U \subset S^1$ containing 1 such that $\psi(b) \notin U$. Hence, $b \notin \psi^{-1}(U)$. We choose K to be a closed disk centered at 0 of radius at least $|b|/\delta$. Hence, $aK \subset \psi^{-1}(U)$ implies $b \notin aK$, meaning $|b| > |a| \cdot |b|/\delta$, so $|a| < \delta$.

Finally, we show Ψ is surjective. From the pairing $\langle , \rangle : k \times \widehat{k} \to S^1$, we have an order-reversing bijection between closed subgroups of $\widehat{k}$ and closed subgroups of k by taking orthogonal complements. Hence, to show $\Psi(k) = \widehat{k}$, it suffices to show $\Psi(k)^\perp = \{0\}$. If $x \in \Psi(k)^\perp$ then $\psi_a(x) = 1$ for all $a \in k$, implying $x = 0$. □

Remark 95. There is a standard nontrivial unitary character ψ for each local field k:

(1) If $k = \mathbb{R}$, we let $\psi(x) := e^{-2\pi i x}$.
(2) If $k = \mathbb{Q}_p$, ψ is defined by $\psi(\mathbb{Z}_p) = 1$ and $\psi(p^{-n}) = e^{2\pi i p^{-n}}$ for all $n \geq 1$.
(3) If $k = \mathbb{F}_p((t))$, define $\psi\left(\sum a_i t^i\right) := e^{2\pi i a_{-1}/p}$ (here we choose a lift of a_{-1} from $\mathbb{F}_p$ to $\mathbb{Z}$).
(4) If k_0 is either $\mathbb{R}, \mathbb{Q}_p$ or $\mathbb{F}_p((t))$ with the corresponding character ψ_0 as above, and k is a finite separable extension of k_0 then let $\psi : k \to S^1$ defined by the composition $k \xrightarrow{\mathrm{Tr}_{k/k_0}} k_0 \xrightarrow{\psi_0} S^1$.

Corollary 96. *We have* $\widehat{\mathbb{R}/\mathbb{Z}} \cong \mathbb{Z}$ *and* $\widehat{\mathbb{Q}_p/\mathbb{Z}_p} \cong \mathbb{Z}_p$ *for prime* p*.*

Proof. It suffices to show that the image of the map $\widehat{\mathbb{R}/\mathbb{Z}} \to \widehat{\mathbb{R}} \xrightarrow{\sim} \mathbb{R}$ is $\mathbb{Z}$. A nontrivial unitary character $f : \mathbb{R}/\mathbb{Z} \to S^1$ induces a nontrivial unitary character $f' : \mathbb{R} \to S^1$ of $\mathbb{R}$ whose kernel contains $\mathbb{Z}$. From previous proposition, it must be of the form $f'(x) = e^{2\pi i a x}$ for some $a \in \mathbb{R}$. Because $f'|_{\mathbb{Z}} = 1$ so $a \in \mathbb{Z}$. Hence, we can define a bijection $\widehat{\mathbb{R}/\mathbb{Z}} \to \mathbb{Z}$ sending f to a.

Similarly, the image of $\widehat{\mathbb{Q}_p/\mathbb{Z}_p} \to \widehat{\mathbb{Q}_p} \xrightarrow{\sim} \mathbb{Q}_p$ is $\mathbb{Z}_p$ because for the standard character ψ of $\mathbb{Q}_p$ defined in Remark 95, $\psi(ax) = 1$ for all $x \in \mathbb{Z}_p$ iff $a \in \mathbb{Z}_p$. □

6.1.2. *Pontryagin dual of adeles.* Recall that the adeles $\mathbb{A}$ of $\mathbb{Q}$ is a locally compact abelian group under addition. From previous propositions, we have the following result

Proposition 97. *We have an isomorphism of topological groups*

$$\begin{aligned} \widehat{\mathbb{A}} &\to \prod_v{}'(\widehat{\mathbb{Q}_v}, \widehat{\mathbb{Q}_v/\mathbb{Z}_v}), \\ \psi &\mapsto (\psi|_{\mathbb{Q}_v}), \\ \prod_v \psi_v &\leftarrow\!\shortmid (\psi_v). \end{aligned}$$

In other words, to give a unitary character ψ *of* $\mathbb{A}$*, it suffices to give a collection* (ψ_v) *of unitary characters of* $\mathbb{Q}_v$ *so that* $\psi_v|_{\mathbb{Z}_v} = 1$ *for almost all places* v *of* $\mathbb{Q}$*.*

Furthermore, we can construct a nontrivial unitary character $\exp_{\mathbb{A}}$ *on* $\mathbb{A}$ *by letting* $\exp_{\mathbb{A}}|_{\mathbb{Q}_v}$ *to be the standard characters on* $\mathbb{Q}_v$ *as in Remark 95. Then*

$$\begin{aligned} \Psi : \mathbb{A} &\to \widehat{\mathbb{A}} \\ a &\mapsto (\exp_{\mathbb{A},a} : x \mapsto \psi(ax)). \end{aligned}$$

is an isomorphism of topological groups.

Sketch. The first isomorphism holds for any restricted product, i.e. if G_v are locally compact abelian groups and H_v is open compact subgroup of G_v then

$$\widehat{\prod{}'(G_v, H_v)} \cong \prod{}' \left(\widehat{G_v}, \widehat{G_v/H_v} \right)$$

with the similar map as defined in the proposition. One can easily show that this map is a bijective group homomorphism. To show it is a homeomorphism requires more work.

For the second isomorphism, the standard character ψ_v on $\mathbb{Q}_v$ induces an isomorphism $\Psi_v : \mathbb{Q}_v \to \widehat{\mathbb{Q}_v}$ that sends $\mathbb{Z}_v$ to $\widehat{\mathbb{Q}_v/\mathbb{Z}_v}$, as shown in corollary 96. Hence, Ψ is precisely the map

$$\mathbb{A} = \prod_v{}'(\mathbb{Q}_v, \mathbb{Z}_v) \xrightarrow{\prod \Psi_v} \prod_v{}'(\widehat{\mathbb{Q}_v}, \widehat{\mathbb{Q}_v/\mathbb{Z}_v}) \xrightarrow{\sim} \widehat{\mathbb{A}}.$$

□

Remark 98. For a general global field k, one can choose a standard character $\exp_{\mathbb{A}_k}$ on $\mathbb{A}_k$ as follows:

(1) If k is a number field, we choose the standard character ψ_v on each k_v as in Remark 95, and choose $\exp_{\mathbb{A}_k} = \prod_v \psi_v$ as the standard character on $\mathbb{A}_k$.

(2) If k is the function field of a curve X over $\mathbb{F}_q$. Let $\Omega_{X/\mathbb{F}_q}$ be the cotangent sheaf, i.e. the sheaf of 1-forms. Let Ω_k be the fiber at the generic point, so Ω_k is a 1-dimensional k-vector space. Define $\Omega_v := \Omega_k \otimes_k k_v$. There is a residue map $\mathrm{Res}_v : \Omega_v \to k_v$ defined as follows: if u is a uniformiser of the closed point v, the residue map is

$$\begin{aligned} \mathrm{Res}_v : \Omega_v = \kappa(v)(\!(u)\!)du &\to \kappa(v) \\ \sum_{i \in \mathbb{Z}} a_i u^i du &\mapsto a_{-1}. \end{aligned}$$

Here $\kappa(v)$ is the residue field at the closed point v. This definition is independent of the choice of uniformiser [9]. A choice of a global 1-form $\omega \in \Omega_k$ gives rise to a unitary character on k_v

$$\psi_v(x) := \exp\left(\frac{2\pi i}{q} \mathrm{Tr}_{\kappa(v)/\mathbb{F}_q} \mathrm{Res}(x\omega) \right)$$

for each closed point v of X. Let $\exp_{\mathbb{A}} = \prod_v \psi_v$ be the standard unitary character on $\mathbb{A}_k$.

Corollary 99. *Let ψ be the standard character on $\mathbb{A}$ as in the previous proposition. Then ψ is trivial on $\mathbb{Q}$ and the isomorphism $\mathbb{A} \cong \widehat{\mathbb{A}}$ defined via ψ gives rise to an isomorphism of topological groups $\mathbb{Q} \cong \widehat{\mathbb{Q} \setminus \mathbb{A}}$.*

Proof. We first recall the definition of $\psi = \prod_v \psi_v$. Here $\psi_\infty : \mathbb{R} \to S^1$ is defined as $\psi_\infty(x) = e^{-2\pi i x}$ and for prime p, if $x = up^n \in \mathbb{Q}_p$ where $u \in \mathbb{Z}_p^\times$, then $\psi_p(x) = 0$ if $n \geq 0$ and $\psi_p(x) = e^{2\pi i p^n (xp^{-n} \mod p^{-n})}$ if $n < 0$. Hence, to show $\psi(x) = 1$ for $x \in \mathbb{Q}$, it suffices to show that if $x = p_1^{k_1} \cdots p_\ell^{k_\ell} \in \mathbb{Q}$ then $x - \sum_{p_i \text{ s.t. } k_i < 0} p_i^{k_i}(xp^{-k_i} \mod p^{-k_i}) \in \mathbb{Z}$, which is true. Thus, ψ is trivial on $\mathbb{Q}$.

Next, we will show $\mathbb{Q} \cong \widehat{\mathbb{Q} \setminus \mathbb{A}}$. Since $\mathbb{Q} \setminus \mathbb{A}$ is compact, $\widehat{\mathbb{Q} \setminus \mathbb{A}}$ is discrete. Under the identification $\psi : \widehat{\mathbb{A}} \cong \mathbb{A}$, $\mathbb{Q} \setminus \Psi(\widehat{\mathbb{Q} \setminus \mathbb{A}})$ is a discrete subgroup of the compact group $\mathbb{Q} \setminus \mathbb{A}$, implying $\mathbb{Q} \setminus \Psi(\widehat{\mathbb{Q} \setminus \mathbb{A}})$ is finite [10].

[9] For formal Laurent series f, g with $\mathrm{ord}(f) \geq 1$ then $\mathrm{Res}((g \circ f)f') = \mathrm{ord}(f)\mathrm{Res}(g)$

[10] We show that a discrete subgroup H of a compact group G has to be finite. Indeed, as H is discrete, there exists an open neighborhood U of 1 so $H \cap U = \{1\}$. This follows $aU \cap H$ is either empty if $a \notin H$ or $\{a\}$ if $a \in H$. Because G is compact, G is a finite union of aU's for $a \in G$, implying H is finite.

On the other hand, $\Psi(\widehat{\mathbb{Q}\setminus\mathbb{A}})$ is a $\mathbb{Q}$-subspace of $\mathbb{A}$, as if $\psi_a|_{\mathbb{Q}} = 1$ then $\psi_{qa}|_{\mathbb{Q}} = 1$ for all $q \in \mathbb{Q}$. This means $\mathbb{Q}\setminus\Psi(\widehat{\mathbb{Q}\setminus\mathbb{A}})$ is a finite $\mathbb{Q}$-vector space. As $\mathbb{Q}$ is infinite so $\mathbb{Q} = \Psi(\widehat{\mathbb{Q}\setminus\mathbb{A}})$, as desired. □

Remark 100. For an arbitrary global field k, one can also show that the standard character $\exp_{\mathbb{A}_k}$ (as defined in Remark 98) is trivial on k, hence inducing an isomorphism $k \cong \widehat{k\backslash\mathbb{A}_k}$ of topological spaces.

6.2. **Fourier transform.** In this subsection, we will discuss Fourier transform on G, in particular when G is $\mathbb{Q}_p, \mathbb{R}$ or $\mathbb{A}_{\mathbb{Q}}$.

If $f \in L^1(G)$, we can define the *Fourier transform* $\widehat{f} : \widehat{G} \to \mathbb{C}$ by

$$\widehat{f}(\chi) := \int_G f(g)\chi(g)dg.$$

One can show $\widehat{f} : \widehat{G} \to \mathbb{C}$ is always continuous. [11]

Under the condition that the function on G is nice enough, we have the following Fourier inversion formula

Theorem 101 (Fourier inversion formula)**.** *Let G be a locally compact abelian group. Let dg be a Haar measure on G. Then there exists a unique Haar measure $d\chi$ on $\widehat{G}$, called the* Plancherel measure*, such that if $f \in L^1(G)$ is such that $\widehat{f} \in L^1(\widehat{G})$ then*

$$f(g) := \int_{\widehat{G}} \widehat{f}(\chi)\overline{\chi(g)}d\chi \tag{3}$$

for almost everywhere g, i.e. there exists a null-set $N \subset G$ such that the above formula holds for all $g \in G \setminus N$.

We refer to [Fol16, p. 111] for the proof of this theorem. Note that under Pontryagin duality, the Fourier inversion formula can be written as $\hat{\hat{f}}(x) = f(-x)$.

Example 102. If G is discrete with the counting measure, the Plancherel measure on the compact group $\widehat{G}$ is the normalised Haar measure so that $\widehat{G}$ has volume 1. Indeed, consider $f \in L^1(G)$ defined as $f(x) = 1$ if $x = 1$ in G and 0 everywhere else. Hence, we find

$$\widehat{f}(\chi) = \sum_{g\in G} f(g)\chi(g) = \chi(1) = 1.$$

By the Fourier inversion formula,

$$1 = f(1) = \int_{\widehat{G}} \widehat{f}(\chi)\overline{\chi(1)}d\chi = d\chi(\widehat{G}).$$

In Theorem 101, the Fourier inversion formula depends on the condition that $\widehat{f} \in L^1(\widehat{G})$. One can define the space of Schwartz-Bruhat functions on a locally compact abelian group so that Fourier transform is an isomorphism on these spaces. In the next subsections, we will focus on defining such functions when G is a local field or G is the adeles. Furthermore, in such cases of G, there is a natural Haar measure on G such that its pushforward via $G \cong \widehat{G}$ (as discussed in the previous section) is the Plancherel measure on $\widehat{G}$ in the Fourier inversion formula.

[11] Some authors, such as [Fol16], define $\widehat{f}$ by taking complex conjugate of $\chi(g)$.

6.2.1. *Fourier transform for local fields.*

Definition 103. *For a local field k, a function $f : k \to \mathbb{C}$ is called a* Schwartz-Bruhat function *if*

- *When $k = \mathbb{R}^n$, f is a C^∞-function whose derivatives are rapidly decreasing [12], i.e. for any $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$, let $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $D^\beta f := \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n} f$, we have*
$$\sup_{x \in \mathbb{R}^n} \left| x^\alpha D^\beta f(x) \right| < \infty.$$
- *When $k = \mathbb{C}^n$, f is viewed as a function on $\mathbb{R}^{2n}$ with rapidly decreasing derivatives.*
- *When k is a nonarchimedean local field, f is a locally constant function of compact support.*

We denote $S(k)$ to be the complex vector space of Schwartz-Bruhat functions on k.

Example 104. If $f(x) \in \mathbb{R}[x_1, \dots, x_n]$ then $f(x)e^{-a|x|^2} \in S(\mathbb{R}^n)$. All compactly supported real C^∞-functions are Schwartz functions.

Example 105. Every Schwartz-Bruhat function $f \in S(\mathbb{Q}_p)$ can be written as $f = \sum_{i=1}^n c_i 1_{a_i + p^{k_i}\mathbb{Z}_p}$ where $a_i \in \mathbb{Q}_p, k_i \in \mathbb{Z}$ and $c_i \in \mathbb{C}$. Indeed, because every open set in $\mathbb{Q}_p$ is a disjoint union of open balls $a + p^k\mathbb{Z}_p$ (for some $a \in \mathbb{Q}_p$ and $k \in \mathbb{Z}$) and that f is compactly supported, the support of f is a finite disjoint union of such open balls. As f is also locally constant, we are done.

Upon identifying k with $\widehat{k}$ via a choice of a standard unitary character as in Remark 95, we can rewrite the Fourier inversion formula as follows

Theorem 106. *Let k be a local field and ψ be the standard unitary character on k as in Remark 95. Under the identification $k \cong \widehat{k}$ via ψ, the Fourier transform*
$$\widehat{f}(y) := \int_k f(x)\psi(xy)dx$$
defines an automorphism of vector spaces on $S(k)$.

There is a unique Haar measure dx on k such that its pushforward via $k \cong \widehat{k}$ is the Plancherel measure on $\widehat{k}$. We call dx the self-dual *Haar measure on k. Under such choice of measure, we have the Fourier inversion formula*
$$f(x) = \int_k \widehat{f}(y)\overline{\psi(xy)}dy.$$
In particular, the self-dual Haar measure on k can be described explicitly as follows:

1. *If $k = \mathbb{R}$ then dx is the Lebesgue measure.*
2. *If $k = \mathbb{C}$ then dx is twice the Lebesgue measure.*
3. *If k is nonarchimedean then dx is the Haar measure for which its ring of integers $\mathcal{O}$ has measure $(\#\mathcal{O}/\mathcal{D})^{-1/2}$, where $\mathcal{D}$ is the different of the field extension $k/\mathbb{Q}_p$ or $k/\mathbb{F}_p(\!(t)\!)$.*

Proof. We will defer the proof of this proposition for the final version of our thesis. At the moment, we will refer to [VR99, p. 300] for further discussions. □

6.2.2. *Fourier transform for adeles.* In this section, we work with the adeles of $\mathbb{Q}$, but the statement will hold for the adeles of any global field k.

Definition 107. *Let k be a global field, a* Schwartz-Bruhat *function $f : \mathbb{A}_k \to \mathbb{C}$ is a finite $\mathbb{C}$-linear combination of $\prod_v f_v : \mathbb{A} \to \mathbb{C}$, where $f_v \in S(\mathbb{Q}_v)$ and $f_v|_{\mathcal{O}_{k_v}} = 1$ for almost all places v of k. We denote $S(\mathbb{A}_k)$ to the space of Schwartz-Bruhat functions on $\mathbb{A}_k$.*

We can describe the Fourier transform on $\mathbb{A}$ in the same way as how we have done for local fields.

[12] in the case where $k = \mathbb{R}^n$, such f is also called Schwartz function

Theorem 108. *We fix a standard unitary character* $\exp_{\mathbb{A}}$ *for* $\mathbb{A}_k$ *as given in Remark 98 and let* dx *to be the self-dual measure on* $\mathbb{A}$ *with respect to* ψ. *For* $f \in S(\mathbb{A})$, *the Fourier transform*

$$\widehat{f}(y) := \int_{\mathbb{A}} f(x) \exp_{\mathbb{A}}(xy) dx$$

defines an isomorphism on $S(\mathbb{A}_k)$. *We also have the Fourier inversion formula* $\hat{\hat{f}}(x) = f(-x)$ *for all* $x \in \mathbb{A}_k$. *In particular, the self-dual Haar measure on* $\mathbb{A}_k$ *is the restricted product of the self-dual Haar measure on* k_v *as described in proposition Theorem 106.*

6.3. **Poisson summation formula.** Consider an exact sequence of locally compact Hausdorff abelian groups

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

where A, B, C are equipped with Haar measures $d\mu_A, d\mu_B, d\mu_C$ that make the following equation holds:

$$\int_B f(b) d\mu_B(b) = \int_A \int_C f(c+a) d\mu_A(a) d\mu_C(c)$$

for all $f \in C_c(B)$.

The Poisson summation formula is essentially the special case of the following result:

Theorem 109. *For any Schwartz-Bruhat function* $f : B \to \mathbb{C}$, *we have*

$$\int_A f(a) d\mu_A(a) = \int_{\widehat{C}} \widehat{f}(\widehat{c}) d\mu_{\widehat{C}}$$

where $\widehat{f} : \widehat{B} \to \mathbb{C}$ *is the Fourier dual of* f, $d\mu_{\widehat{C}}$ *is the dual Haar measure on* $\widehat{C}$.

Sketch of proof. Define $F(x) = \int_A f(x+a) d\mu_A$ as a function on C. By Fourier inversion formula, we have

$$\begin{aligned}
\widehat{F}(\chi_C) &= \int_C F(c) \overline{\chi_C(c)} d\mu_C(c), \\
&= \int_C \int_A f(c+a) \overline{\chi_C(c)} d\mu_C(c) d\mu_A(a), \\
&= \int_B f(b) \overline{\chi_C(b)} d\mu_B(b), \\
&= \widehat{f}(\chi_C).
\end{aligned}$$

Again, by Fourier inversion formula, we find

$$F(c) = \int_{\widehat{C}} \widehat{F}(\chi_C) \overline{\chi_C(c)} d\mu_{\widehat{C}}(\chi_C)$$

which may be written as

$$\int_A f(c+a) d\mu_A(a) = \int_{\widehat{C}} \widehat{f}(\chi_C) \overline{\chi_C(c)} d\mu_{\overline{C}}(\chi_C).$$

By letting $c = 0$, we get the desired identity. □

In the special case where L is a lattice in B (i.e. L is discrete and B/L is compact) then the dual space $L^{\perp} = \widehat{B/L}$ is a lattice inside $\widehat{B}$. From Example 102, the counting measure on $\widehat{B/L}$ is dual to the normalised Haar measure on B/L, giving

$$\sum_{x \in L} f(x) = \frac{1}{\mu_B(B/L)} \sum_{y \in L^{\perp}} \widehat{f}(y)$$

The measure on B is often chosen so that $\mu(B/L) = 1$.

Example 110. From the exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{Z} \setminus \mathbb{R} \to 0$ and that $\widehat{\mathbb{Z} \setminus \mathbb{R}} \cong \mathbb{Z}$, we find

$$\sum_{x \in \mathbb{Z}} f(x) = \frac{1}{\mu_{\mathbb{R}}(\mathbb{Z} \setminus \mathbb{R})} \sum_{x \in \mathbb{Z}} \widehat{f}(x).$$

Applying the above equality to $\widehat{f}$ yields

$$\sum_{x \in \mathbb{Z}} \widehat{f}(x) = \frac{1}{\mu_{\mathbb{R}}(\mathbb{Z} \setminus \mathbb{R})} \sum_{x \in \mathbb{Z}} f(x).$$

Combining these two identities, we find $\mu_{\mathbb{R}}(\mathbb{Z} \setminus \mathbb{R}) = 1$.

The same exact argument for $0 \to \mathbb{Q} \to \mathbb{A} \to \mathbb{Q} \setminus \mathbb{A} \to 0$ shows $\mu_{\mathbb{A}}(\mathbb{Q} \setminus \mathbb{A}) = 1$.

7. SL_2

In this section, we will describe SL_2 as a linear algebraic group over a field k. We will then define and derive a nonvanishing, left-invariant global top form for SL_2.

7.1. **Affine algebraic group** SL_2**.** The affine algebraic group SL_2 over a field k is the morphism of affine schemes $\operatorname{Spec}\mathcal{O}(\mathrm{SL}_2) \to \operatorname{Spec} k$ where

$$\mathcal{O}(\mathrm{SL}_2) = k[x_{11}, x_{12}, x_{21}, x_{22}]/(x_{11}x_{22} - x_{21}x_{12} - 1).$$

The k-algebra $\mathcal{O}(\mathrm{SL}_2)$ is equal to the ideal generated by x_{11}, x_{12} as for any $f \in \mathcal{O}(\mathrm{SL}_2)$ then $f = f(x_{11}x_{22} - x_{21}x_{12}) \in (x_{11}, x_{12})$. Therefore, $\operatorname{Spec}\mathcal{O}(\mathrm{SL}_2) = D(x_{11}) \cup D(x_{12})$ where for $f \in \mathcal{O}(\mathrm{SL}_2), D(f) := \{\mathfrak{p} \in \operatorname{Spec}\mathcal{O}(\mathrm{SL}_2) : f \notin \mathfrak{p}\}$ is the distinguished open set of $\operatorname{Spec}\mathcal{O}(\mathrm{SL}_2)$.

For a k-algebra R, the R-points of SL_2, denoted by $\mathrm{SL}_2(R)$ is the group $\operatorname{Hom}_{k-\mathrm{alg}}(\mathcal{O}(\mathrm{SL}_2), R)$, which can be identified with $\left\{\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{pmatrix} : x_{ij} \in R, x_{11}x_{22} - x_{12}x_{21} = 1\right\}$ with the usual group structure.

For a ring R, we use $\mathrm{SL}_{2,R}$ to denote SL_2 over R.

7.2. **Lie algebra.** In this section, we define the Lie algebra of SL_2. We have a projection $k[\epsilon]/(\epsilon^2) \to k$ sending $a + \epsilon b$ to a. For k-algebra R, we define the *Lie algebra* of SL_2 over R to be

$$\operatorname{Lie}(\mathrm{SL}_2)(R) := \ker(\mathrm{SL}_2(R[\epsilon]/(\epsilon^2)) \to \mathrm{SL}_2(R)) \tag{4}$$

In particular, one can describe elements in $\operatorname{Lie}(\mathrm{SL}_2)(k)$ as 2-by-2 matrices of determinant 1, with entries over $k[\epsilon]/(\epsilon^2)$, such that by letting $\epsilon \mapsto 0$, we get the identity matrix. Concretely, elements of $\operatorname{Lie}(\mathrm{SL}_2)$ are of the form $\begin{pmatrix} 1+\epsilon a_{11} & \epsilon a_{12} \\ \epsilon a_{21} & 1+\epsilon a_{22}\end{pmatrix} = I_2 + \epsilon\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}$ such that $a_{11} + a_{22} = 0$, which we can identify with $\mathfrak{sl}_2(k)$, a k-vector space of 2-by-2 matrices having trace 0.

In general, for an affine algebraic group G, it is more subtle to see the Lie algebra structure from the definition (4) of tangent space. However, we can embed G into GL_n and the Lie algebra structure of $\operatorname{Lie}(G)$ is induced from this embedding.

We find that for a k-algebra R, $\operatorname{Lie}(\mathrm{SL}_{2,R}) \cong R \otimes_k \operatorname{Lie}(\mathrm{SL}_2)$ as R-modules.

The dual $\operatorname{Lie}(\mathrm{SL}_2)^*$ is a k-module generated by dx_{ij}, for $1 \le i, j \le 2$, modulo the relation $dx_{11} + dx_{22} = 0$.

7.3. **Differential forms.** To define the cotangent sheaf of SL_2 over k, we first need to define the *module of relative differentials* $\Omega_{\mathcal{O}(\mathrm{SL}_2)/k}$. It is a $\mathcal{O}(\mathrm{SL}_2)$-module equipped with a k-derivation $d : \mathcal{O}(\mathrm{SL}_2) \to \Omega_{\mathcal{O}(\mathrm{SL}_2)/k}$ that is universal as initial object among $\mathcal{O}(\mathrm{SL}_2)$-modules M equipped with k-derivation $d : \mathcal{O}(\mathrm{SL}_2) \to M$. Concretely, $\Omega_{\mathcal{O}(\mathrm{SL}_2)/k}$ is a $\mathcal{O}(\mathrm{SL}_2)$-module generated by dx_{ij} for $x_{ij} \in \mathcal{O}(\mathrm{SL}_2), 1 \le i, j \le 2$ quotient out by the relation $x_{11}dx_{22} + x_{22}dx_{11} - x_{12}dx_{21} - x_{21}dx_{12}$. The map $d : \mathcal{O}(\mathrm{SL}_2) \to \Omega_{\mathcal{O}(\mathrm{SL}_2)/k}$ is the obvious one, as suggested by the notation.

We define the *cotangent sheaf* $\Omega_{\mathrm{SL}_2/k}$ [13] to be the sheaf of $\mathcal{O}_{\mathrm{SL}_2}$-modules associated to $\mathcal{O}(\mathrm{SL}_2)$-module $\Omega_{\mathcal{O}(\mathrm{SL}_2)/k}$. Concretely, for $f \in \mathcal{O}(\mathrm{SL}_2)$, its section over distinguished open $D(f) = \{\mathfrak{p} \in \operatorname{Spec}\mathcal{O}(\mathrm{SL}_2) : f \notin \mathfrak{p}\}$ in $\operatorname{Spec}\mathcal{O}(\mathrm{SL}_2)$ is the localisation of $\Omega_{\mathcal{O}(\mathrm{SL}_2)/k}$ at f. Note also that the global section of $\Omega_{\mathrm{SL}_2/k}$ is precisely $\Omega_{\mathcal{O}(\mathrm{SL}_2)/k}$.

Proposition 111. *(a) The cotangent sheaf $\Omega_{\mathrm{SL}_2,k}$ is a vector bundle of rank 3, hence a cotangent bundle.*

(b) The fiber of $\Omega_{\mathrm{SL}_2,k}$ at a point $\mathfrak{p} \in \operatorname{Spec}\mathcal{O}(\mathrm{SL}_2)$ is $\Omega_{\mathcal{O}(\mathrm{SL}_2)/k} \otimes_{\mathcal{O}(\mathrm{SL}_2)} \kappa(\mathfrak{p})$, where $\kappa(\mathfrak{p}) = \mathcal{O}(\mathrm{SL}_2)_{\mathfrak{p}}/\mathfrak{m} \cong k$ and $\mathfrak{m}$ is the maximal ideal of the local ring $\mathcal{O}(\mathrm{SL}_2)_{\mathfrak{p}}$. The fiber of $\Omega_{\mathrm{SL}_2/k}$ at $\mathfrak{p}$ is isomorphic as k-vector space to the cotangent space $\mathfrak{m}/\mathfrak{m}^2$.

[13] notice that there is a subtlety in our choice of notations, where $\Omega_{\mathcal{O}(\mathrm{SL}_2)/k}$ is different from $\Omega_{\mathrm{SL}_2/k}$

A *differential* 1*-form over open* U in SL_2 is a section of $\Omega_{\mathrm{SL}_2/k}$ over U. For example, in $\Gamma(\Omega_{\mathrm{SL}_2/k}, D(x_{11})) = \left(\Omega_{\mathcal{O}(\mathrm{SL}_2)/k}\right)_{x_{11}}$, we have

$$dx_{22} = \frac{x_{11}dx_{22}}{x_{11}} = \frac{1}{x_{11}}(x_{12}dx_{21} + x_{21}dx_{12} - x_{22}dx_{11}). \tag{5}$$

Therefore, any differential 1-form over $D(x_{11})$ can be written as

$$\frac{1}{x_{11}^k}(f_{12}dx_{12} + f_{21}dx_{21} + f_{11}dx_{11}),$$

where $f_{ij} \in \mathcal{O}(\mathrm{SL}_2)$.

We define the *canonical sheaf* ω_{SL_2} to be $\omega_{\mathrm{SL}_2} = \bigwedge^3 \Omega_{\mathrm{SL}_2/k}$. Its sections over open U of $\operatorname{Spec}\mathcal{O}(\mathrm{SL}_2)$ are called *top (dimensional) forms of* SL_2 *over* U. For example, its sections over $D(x_{11})$ form a $\mathcal{O}(\mathrm{SL}_2)_{x_{11}}$-module generated by $dx_{11} \wedge dx_{12} \wedge dx_{21}$. If ω is a top form of SL_2 over U then we say ω is *nowhere vanishing* if $\omega_x \in (\omega_{\mathrm{SL}_2})_x$ is nonzero for all $x \in U$.

7.4. **Left-invariant differential form.** We consider a k-algebra isomorphism

$$\begin{aligned} L_a : \mathcal{O}(\mathrm{SL}_2) &\to \mathcal{O}(\mathrm{SL}_2), \\ x_{ij} &\mapsto a_{i1}x_{1j} + a_{i2}x_{2j},\ 1 \le i,j \le 2. \end{aligned}$$

corresponding to left-multiplication by $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{SL}_2(k)$. This induces an isomorphism of $\mathcal{O}(\mathrm{SL}_2)$-modules

$$\begin{aligned} L_a : \Omega_{\mathcal{O}(\mathrm{SL}_2)/k} &\to \Omega_{\mathcal{O}(\mathrm{SL}_2)/k} \\ f dx_{ij} &\mapsto L_a(f)(a_{i1}dx_{1j} + a_{i2}dx_{2j}), \end{aligned}$$

for any $f \in \mathcal{O}(\mathrm{SL}_2)$, hence, an isomorphism of sheaves of $\mathcal{O}_{\mathrm{SL}_2}$-modules $L_a : \omega_{\mathrm{SL}_2} \to \omega_{\mathrm{SL}_2}$.

A top form ω over open set U is called *left-invariant* if $L_a\omega = \omega$ for any $a \in \mathrm{SL}_2(k)$.

Proposition 112. *There is a unique, nowhere vanishing, left-invariant, global top form for* SL_2 *up to scalar over* $k^\times$*.*

Proof. We first determine all left-invariant top forms ω_{11} over $D(x_{11})$. From previous section, we can write ω_{11} over $D(x_{11})$ as $f dx_{11} \wedge dx_{12} \wedge dx_{21}$ where $f \in \mathcal{O}(\mathrm{SL}_2)_{x_{11}}$. It follows that over $D(x_{11})$, we have

$$\begin{aligned} L_a\omega_{11} &= L_a(f) d(a_{11}x_{11} + a_{12}x_{21}) \wedge d(a_{11}x_{12} + a_{12}x_{22}) \wedge d(a_{21}x_{11} + a_{22}x_{21}), \\ &= \frac{L_a(f)}{x_{11}}(a_{11}x_{11} + a_{12}x_{21}) dx_{11} \wedge dx_{12} \wedge dx_{21}. \end{aligned}$$

Therefore, for ω_{11} to be left-invariant, we must have $x_{11}f = (a_{11}x_{11} + a_{12}x_{21})L_a(f)$ for any $f \in \mathcal{O}(\mathrm{SL}_2)_{x_{11}}$ and any $a_{ij} \in k$ such that $a_{11}a_{22} - a_{12}a_{21} = 1$. This implies $f = Cx_{11}^{-1}$ for $C \in k$. Thus, $\omega_{11} = Cx_{11}^{-1}dx_{11} \wedge dx_{12} \wedge dx_{21}$ for some $C \in k^\times$.

Similarly, we also can find a left-invariant top form ω_{12} over $D(x_{12})$ to be $\omega_{12} = C'x_{12}^{-1}dx_{11} \wedge dx_{12} \wedge dx_{22}$ for some $C' \in k^\times$.

As $\operatorname{Spec}\mathcal{O}(\mathrm{SL}_2) = D(x_{11}) \cup D(x_{12})$, a global left-invariant top form $\omega \in \Gamma(\Omega_{\mathrm{SL}_2/k}, \operatorname{Spec}\mathcal{O}(\mathrm{SL}_2))$, if exists, must correspond to ω_{11} and ω_{12} when restricting to $D(x_{11})$ and $D(x_{12})$, respectively. Hence, to find such global top form, it suffices to find $C, C' \in k^\times$ such that $\omega_{11} = \omega_{12}$ on $D(x_{11}) \cap D(x_{12}) = D(x_{11}x_{12})$. Indeed, on $D(x_{11}x_{12})$, dx_{22} can be written as in (5), hence $\omega_{12} = \frac{C'x_{12}}{x_{11}x_{12}}dx_{11} \wedge dx_{12} \wedge dx_{21}$. Therefore, $\omega_{11} = \omega_{12}$ gives $C = C'$. Thus, SL_2 has a unique , nowhere-vanishing, left-invariant global top form up to scalar over $k^\times$. □

Next, we will identify left-invariant global top forms over k with $\bigwedge^3 \mathrm{Lie}(\mathrm{SL}_2)(k)^*$, following [BLR90, §4.2]. Indeed, the unit element in the group structure of SL_2 corresponds to the k-algebra morphism $\varepsilon : \mathcal{O}(\mathrm{SL}_2) \to k$ sending x_{ij} to 1 if $1 \leq i = j \leq 2$ and 0 everywhere else. This then corresponds to a morphism $\varepsilon : \operatorname{Spec} k \to \operatorname{Spec} \mathcal{O}(\mathrm{SL}_2)$ of affine scheme. Therefore, one can pullback sheaf $\Omega_{\mathrm{SL}_2/k}$ of $\mathcal{O}_{\mathrm{SL}_2}$-module via ε to get a sheaf $\varepsilon^*\Omega_{\mathrm{SL}_2/k}$ of $\mathcal{O}_{\operatorname{Spec} k}$-modules, which is just a k-module $k \otimes_{\mathcal{O}(\mathrm{SL}_2)} \Omega_{\mathcal{O}(\mathrm{SL}_2)/k}$. We note that this k-module is isomorphic to $\mathrm{Lie}(\mathrm{SL}_2)(k)^*$.

On the other hand, via the structural morphism $p : \operatorname{Spec} \mathcal{O}(\mathrm{SL}_2) \to \operatorname{Spec} k$, we have a canonical isomorphism $p^*\varepsilon^*\Omega_{\mathrm{SL}_2/k} \xrightarrow{\sim} \Omega_{\mathrm{SL}_2/k}$ that is obtained by extending sections in $\varepsilon^*\Omega_{\mathrm{SL}_2/k} = \mathrm{Lie}(\mathrm{SL}_2)(k)^*$ to left-invariant sections in $\Omega_{\mathrm{SL}_2/k}$ (see [BLR90, page 102]). Thus, the k-module $\bigwedge^3 \mathrm{Lie}(\mathrm{SL}_2)(k)^*$ is identified with the k-module of left-invariant global top forms.

7.5. **Adjoint map.** Given a k-algebra R, an affine algebraic group G and its Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$, we define the *adjoint representation* $\mathrm{Ad} : G(R) \to \mathrm{Aut}(\mathfrak{g}(R))$ to be $\mathrm{Ad}(g)x = gxg^{-1}$ where $x \in \mathrm{Lie}(G)(R) \subset G(R[\varepsilon]/(\varepsilon^2))$.

In particular, if $\omega \in \bigwedge^3 \mathfrak{g}(R)^*$ then

$$\mathrm{Ad}(g)\omega = \det\left(\mathrm{Ad}(g) : \mathfrak{g}(R) \to \mathfrak{g}(R)\right)\omega.$$

8. The Tamagawa measure on algebraic groups

For an algebraic group G over a global field k, there exists a left-invariant (or right-invariant) volume form ω over k [14] that is unique up to scalar over $k^\times$. Following § 5.5.1, one can construct a Radon measure $|\omega|_v$ on $G(k_v)$ for every place v of k. In particular, this is a left-invariant (respectively, right-invariant) Haar measure if ω is left-invariant (respectively, right-invariant). In this section, we will show that when G is connected semisimple, G admits an adelic Haar measure coming from $|\omega|_v$'s (i.e. see § 5.5.2 for the definition). Furthermore, this is a canonical adelic Haar measure on $G(\mathbb{A}_k)$, and we called it the Tamagawa measure of $G(\mathbb{A}_k)$. We will then define the Tamagawa number of G and state the Weil's conjecture on the Tamagawa numbers.

8.1. About normalisation of adelic measure and Tamagawa measure on $\mathbb{A}^n$.

For a nonarchimedean local field k_v, we have chosen two normalisations for a Haar measure on k_v, one in § 5.5.1 where we require $\mathcal{O}_v$ having volume 1, and one in Theorem 106 as a self-dual Haar measure with respect to Fourier transform. In this section, we will would like to explain the relation between these two normalisations in defining adelic measure on $\mathbb{A}$.

Proposition 113. *Let ρ_k be the normalisation in our definition of adelic measure as in § 5.5.3. With respect to the Haar measures μ_v on nonarchimedean local fields k_v such that $\mathcal{O}_v$ having volume 1 (and to the normal Haar measures μ_v on archimedean local fields), we obtain an adelic Haar measure $\mu_k = \prod_v{}' \mu_v$ on $\mathbb{A}_k$. This measure induces an $\mathbb{A}_k$-invariant measure on $k\backslash\mathbb{A}_k$ where $\mu_k(k\backslash\mathbb{A}_k) = \rho_k$.*

We refer to [Wei82, p. 12] for the proof of this proposition.

This proposition says that the normalised adelic measure $\rho_k^{-n}\prod_v{}'\mu_v$ as in § 5.5.3 is the self-dual Haar measure on $\mathbb{A}_k^n$ with respect to Fourier transform (see Theorem 108), as both give volume 1 for $k^n\backslash\mathbb{A}_k^n$ (with respect to self-dual Haar measure on $\mathbb{A}_k$, the volume of $k\backslash\mathbb{A}_k$ being 1 is proved in § 6.3 using Poisson summation formula) and both are Haar measures on $\mathbb{A}_k^n$. We call this the *Tamagawa measure* $\mu_{\mathbb{G}_a^n,k}$ on $\mathbb{A}_k^n$ and the volume of $k^n\backslash\mathbb{A}_k^n$ with respect to this measure is called the *Tamagawa number* of $\mathbb{G}_{a,k}^n$, denoted $\tau_k(\mathbb{G}_a^n)$, and we know $\tau_k(\mathbb{G}_a^n) = 1$.

Furthermore, we can restate our normalised adelic measure for a smooth affine scheme X of finite type over k as in § 5.5.3 in terms of self-dual Haar measures on each local field k_v. This is done by simply removing the normalisation $\rho_k^{-\dim X}$.

Example 114. When $k = \mathbb{Q}$, our two normalisations of Haar measure on $\mathbb{Q}_p$ coincide. Hence, the volume of $\mathbb{Q}\backslash\mathbb{A}$ is 1 because the quotient $\mathbb{Q}\backslash\mathbb{A}$ has a fundamental domain

$$[0,1) \times \prod_p \mathbb{Z}_p,$$

giving the volume $\mu_{\mathbb{R}}([0,1)) \times \prod_p \mu_{\mathbb{Q}_p}(\mathbb{Z}_p) = 1$ for $\mathbb{Q}\backslash\mathbb{A}$.

8.2. Tamagawa measure and Tamagawa number on semisimple groups.

Let G be a connected semisimple algebraic group over a global field k. In this section, we show that G admits an adelic Haar measure There exists a left-invariant algebraic differential form ω of top degree (i.e. a gauge form) of G over k. This induces a left-invariant Haar measure $|\omega|_v$ on $G(k_v)$ with respect to the self-dual Haar measure on k_v. In this section, we show that G admits an adelic Haar measure.

Firstly, for semisimple G (or even for unipotent or reductive G), the Haar measure on G is unimodular (i.e. both left and right invariance) due to the following proposition

Proposition 115. *The modular quasicharacter of $G(k_v)$ is*

$$\delta_{G(k_v)}(g) = \left|\det\left(\mathrm{Ad}(g) : \mathfrak{g}(k_v) \to \mathfrak{g}(k_v)\right)\right|_v.$$

[14] many references refer to this as a gauge form

Proof. From § 7.5, we know $\mathrm{Ad}(g)\omega = \det(\mathrm{Ad}(g) : \mathfrak{g}(k_v) \to \mathfrak{g}(k_v))\,\omega$. Therefore, by Change of variables formula in Theorem 67, we find that $\mathrm{Ad}(g) : G(k_v) \to G(k_v)$ induces a new left Haar measure $d|\omega|_v(ghg^{-1})$ on $G(k_v)$ so that

$$d|\omega|_v(ghg^{-1}) = |\det(\mathrm{Ad}(g) : \mathfrak{g}(k_v) \to \mathfrak{g}(k_v))|_v \, d|\omega|_v.$$

As $d|\omega|_v$ is left Haar measure so from Proposition 40, we find

$$d|\omega|_v(ghg^{-1}) = d|\omega|_v(hg^{-1}) = \delta_{G(k_v)}(g) d|\omega|_v.$$

This gives $\delta_{G(k_v)}(g) = |\det(\mathrm{Ad}(g) : \mathfrak{g}(k_v) \to \mathfrak{g}(k_v))|_v$, as desired. □

Proposition 116. *G admits an adelic measure.*

Sketch. We follow the proof of [Vos98, p. 136]. To show G admits an adelic measure, we need to check the criteria in in § 5.5.2. Let S be a finite set of places of k containing the archimedean places such that there is a smooth group scheme $\overline{G}$ over $\mathcal{O}_S$ with generic fiber G. We want to show

$$\prod_{v \notin S} \frac{|\overline{G}(\kappa(v))|}{|\kappa(v)|^n},$$

converges absolutely, where $\kappa(v)$ is the residue field of k_v, n is the dimension of G.

Steinberg in [SR68, §11.16] gave a general formula for $\overline{G}(\kappa(v))$ (see [Oes84, §I.1.6] for a nicer formulation of this formula). The formula roughly says the following: Let G be a connected semisimple algebraic group over $\mathbb{F}_q$, let B be a Borel subgroup of G containing a maximal torus T. Let $\hat{T} := \mathrm{Hom}(T, \mathbb{G}_m)$. The Galois group $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ acts on the $\mathbb{Q}$-vector space $E = \hat{T}(\overline{\mathbb{F}_q}) \otimes \mathbb{Q}$ by $\sigma(\chi)(x) := \sigma(\chi(\sigma^{-1}(x)))$ for $\sigma \in \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q), \chi \in \hat{T}(\overline{\mathbb{F}_q}), x \in T(\overline{\mathbb{F}_q})$. The Weyl group $W = N_G(T(\overline{\mathbb{F}_q}))/T(\overline{\mathbb{F}_q})$ acts on V via the conjugation action on $T(\overline{\mathbb{F}_q})$. Thus, we obtain an action of $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \ltimes W$ on E, hence an action on the symmetric algebra $S(E)$ of E. A theorem of Chevalley-Shephard-Todd showed that the W-invariant algebra $S(E)^W$ is a polynomial algebra generated by $\ell = \dim T$ homogeneous algebraically independent polynomials of degrees a_i's for $1 \le i \le \ell$. The Steinberg's formula implies that

$$q^{-\ell} \prod_{i=1}^{\ell} (1 - q^{-a_i}) \le |G(\mathbb{F}_q)| \le q^{-\ell} \prod_{i=1}^{\ell} (1 + q^{-a_i}).$$

One can show that $a_i \ge 2$ by relating the a_i's with the Betti numbers b_i of the maximal compact subgroup of $G(\mathbb{C})$ via the formula

$$\sum_{i=0}^{\infty} b_i t^i = \prod_{i=1}^{\ell} (1 + t^{2a_i - 1})$$

and knowing that $b_1 = b_2 = 0$. □

From this proposition, we know that if ω is nowhere-vanishing, left-invariant global algebraic top form (i.e. a gauge form) over k of G inducing Haar measures $|\omega|_v$ on $G(k_v)$, $\prod_v{}' |\omega|_v$ defines a Haar measure on $G(\mathbb{A}_k)$. We call this the *Tamagawa measure* $\mu_{G,k}$ on $G(\mathbb{A}_k)$. This measure satisfies the following nice property

Proposition 117. *The definition of the Tamagawa measure on $G(\mathbb{A})$ does not depend on the choice of a nowhere-vanishing, left-invariant global algebraic top form on G over k.*

Proof. From Proposition 112, any left-invariant, nowhere vanishing global top form ω of G over k is $c\omega$ for some $c \in k^\times$.

Therefore, if $\omega' = c\omega$ is another choice of a top form on G over k, from **??**, the corresponding Haar measure on $G(k_v)$ is $|c|_v \mu_v$ for each place v of k, where μ_v the Haar measure on $G(k_v)$

corresponding to ω. By similar construction, we denote μ'_{Tam} to be the restricted product measure on $G(\mathbb{A})$ corresponding to ω.

Let S be a finite set of places containing the archimedean places so that $\overline{G}$ is a smooth group scheme over $\mathcal{O}_S := \prod_{v \notin S} \mathcal{O}_v$ with generic fiber G. Consider an open subset $U = \prod_{v \in S} G(k_v) \times \prod_{v \notin S} \overline{G}(\mathcal{O}_v)$ of $G(\mathbb{A})$. By construction, the restriction of μ_{Tam} and μ'_{Tam} to U is the product measure. On the other hand, by the product formula $\prod_v |c|_v = 1$ so $\mu_{\mathrm{Tam}} = \mu'_{\mathrm{Tam}}$ on U. By the uniqueness of Haar measure, $\mu_{\mathrm{Tam}} = \mu'_{\mathrm{Tam}}$ on $G(\mathbb{A})$. □

As $G(k)$ is a discrete subgroup of $G(\mathbb{A}_k)$, the Tamagawa measure induces a $G(\mathbb{A}_k)$-invariant measure on $G(k)\backslash G(\mathbb{A}_k)$. Using reduction theory as in [PR94, §5.3], one can show that the volume of $G(k)\backslash G(\mathbb{A}_k)$ is finite, and we call it the *Tamagawa number* of G over k, denoted by $\tau_k(G)$.

Proposition 118. *All groups below will be connected semisimple.*

(1) For algebraic groups G_1, G_2 over k then $\tau_k(G_1 \times G_2) = \tau_k(G_1)\tau_k(G_2)$ and $\tau_k(G_1 \ltimes G_2) = \tau_k(G_1)\tau_k(G_2)$.

(2) Let G be an algebraic group over a finite separable extension l of k, then $\tau_l(G) = \tau_k(Res^l_k G)$, where Res^l_k is the Weil restriction of scalars.

(3) If G splits over k, for an isogeny (i.e. surjective map with finite kernel) $f : G' \to G$ of G then $\tau_k(G) = \tau_k(G') \cdot |\ker f|$.

Proof. For (1), we refer to [Wei82, p. 27] or [Igu78, p. 121].

For (2), this follows from § 5.5.3.

For (3), we refer to [Ono65, Theorem 2.1.1]. □

8.3. **Weil's conjecture on Tamagawa numbers.** We restate Weil's conjecture on Tamagawa numbers.

Theorem 119 ([Wei95])**.** *Let G be a connected simply connected semisimple linear algebraic group over a global field k, then $\tau_k(G) = 1$.*

9. Tamagawa number of SL_2

In this section, we will give a detailed proof of the following theorem

Theorem 120. *$\tau(\mathrm{SL}_{2,\mathbb{Q}}) = 1$.*

9.1. Approximation theorem for SL_2 over $\mathbb{Q}$.

For an affine algebraic group G over $\mathbb{Q}$. We say G satisfies strong approximation with respect to a finite set S of places of $\mathbb{Q}$ if $G(\mathbb{Q})$ is dense in $G(\mathbb{A}^S)$. We then have the following results. We refer the reader to [Rap14] for more discussion.

Theorem 121 (Strong approximation theorem)**.** *Let G be a semisimple and simply connected linear algebraic group G over $\mathbb{Q}$. Then, for any nonempty finite set S of places of $\mathbb{Q}$, $G(\mathbb{A})$ is dense in $G(\mathbb{A}^S)$.*

Corollary 122. *If an affine algebraic group G over $\mathbb{Q}$ satisfies the strong approximation theorem with respect to a finite set $S = \{\infty\}$ of places of $\mathbb{Q}$ then*

(1) $|G(\mathbb{Q}) \setminus G(\mathbb{A}^\infty)/K^\infty| = 1$ for any compact open subgroup K^∞ of $G(\mathbb{A}^\infty)$.

(2) If $\Gamma = G(\mathbb{Q}) \cap K^\infty$ then by embedding $G(\mathbb{R})$ to the infinite component of $G(\mathbb{A})$, we have a homeomorphism

$$\Gamma \setminus G(\mathbb{R}) \to G(\mathbb{Q}) \setminus G(\mathbb{A})/K^\infty.$$

The proof of this corollary is similar to the proof of corollary 73 for $\mathbb{G}_a$. We will give a proof of strong approximation theorem for SL_2 over $\mathbb{Q}$.

Proposition 123 (Strong approximation theorem for SL_2)**.** *For any non-empty finite set S of places of $\mathbb{Q}$, $\mathrm{SL}_2(\mathbb{Q})$ is dense in $\mathrm{SL}_2(\mathbb{A}^S)$.*

Proof. If Z is the closure of $\mathrm{SL}_2(k)$ in $\mathrm{SL}_2(\mathbb{A}^S)$, then Z is a subgroup of $\mathrm{SL}_2(\mathbb{A}^S)$. It suffices to prove that Z contains $\mathrm{SL}_2(\mathbb{Q}_v)$ for every $v \notin S$. Indeed, if such a condition holds then the subgroup Z will contain $\prod_{v\in S'} \mathrm{SL}_2(\mathbb{Q}_v) \times \prod_{v\notin S\cup S'} \mathrm{SL}_2(\mathbb{Z}_v)$ where S' is any finite set of places of $\mathbb{Q}$ disjoint from S. As this exhausts $\mathrm{SL}_2(\mathbb{A}^S)$, we find $Z = \mathrm{SL}_2(\mathbb{A}^S)$.

To show $\mathrm{SL}_2(\mathbb{Q}_v) \subset Z$, note that $\mathrm{SL}_2(\mathbb{Q}_v)$ is generated by $U^+(\mathbb{Q}_v) = \left\{\begin{pmatrix} 1 & * \\ 0 & 1\end{pmatrix}\right\}$ and $U^-(\mathbb{Q}_v) = \left\{\begin{pmatrix} 1 & 0 \\ * & 1\end{pmatrix}\right\}$ so it suffices to show Z contains $U^\pm(\mathbb{Q}_v)$. By definition, Z contains the closure of $U^\pm(\mathbb{Q})$. As $U^+ \cong \mathbb{G}_a$ so by the strong approximation theorem for $\mathbb{G}_a$, the closure of $U^+(\mathbb{Q})$ in $\mathrm{SL}_2(\mathbb{A}^S)$ is $U^+(\mathbb{A}^S)$, implying Z contains $U^+(\mathbb{Q}_v)$ for all $v \notin S$. □

Corollary 124. *(a) We have $\mathrm{SL}_2(\mathbb{A}^\infty) = \mathrm{SL}_2(\mathbb{Q})\,\mathrm{SL}_2(\widehat{\mathbb{Z}})$ and*

$$\mathrm{SL}_2(\mathbb{A}) = \mathrm{SL}_2(\mathbb{Q})\left(\mathrm{SL}_2(\mathbb{R}) \times \prod_p \mathrm{SL}_2(\mathbb{Z}_p)\right).$$

Note that $\mathrm{SL}_2(\mathbb{Q})$ embeds diagonally into $\mathrm{SL}_2(\mathbb{A})$ while for $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{Q}_p)$, each is embedded into its p-component in $\mathrm{SL}_2(\mathbb{A})$.

(b) We have

$$\mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R}) \cong \mathrm{SL}_2(\mathbb{Q}) \setminus \mathrm{SL}_2(\mathbb{A})/\mathrm{SL}_2(\widehat{\mathbb{Z}})$$

so

$$(\mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R})) \times \prod_p \mathrm{SL}_2(\mathbb{Z}_p) \cong \mathrm{SL}_2(\mathbb{Q}) \setminus \mathrm{SL}_2(\mathbb{A})$$

as topological spaces.

One can repeat the proof of corollary 73 to prove this corollary.

9.2. **Tamagawa number for SL_2 over $\mathbb{Q}$.** Because $\mathrm{SL}_2(\mathbb{A})$ is unimodular and $\mathrm{SL}_2(\mathbb{Q})$ is a discrete closed subgroup of $\mathrm{SL}_2(\mathbb{A})$, from § 3.4.2, μ_{Tam} induces a $\mathrm{SL}_2(\mathbb{A})$-invariant measure on the quotient $\mathrm{SL}_2(\mathbb{Q}) \setminus \mathrm{SL}_2(\mathbb{A})$. The volume of $\mathrm{SL}_2(\mathbb{Q}) \setminus \mathrm{SL}_2(\mathbb{A})$ is then called the *Tamagawa number of* SL_2 *over* $\mathbb{Q}$, denoted by $\tau(\mathrm{SL}_{2,\mathbb{Q}})$.

By the construction of the Tamagawa measure on $\mathrm{SL}_2(\mathbb{A})$ and by corollary 124, we obtain

$$\tau(\mathrm{SL}_{2,\mathbb{Q}}) = \mu_{\mathrm{SL}_2(\mathbb{R}),\omega}(\mathrm{SL}_2(\mathbb{Q}) \setminus \mathrm{SL}_2(\mathbb{R})) \times \prod_{\mathrm{SL}_2(\mathbb{Q}_p),\omega} \mu_p(\mathrm{SL}_2(\mathbb{Z}_p))$$

where ω is a choice of a volume form over $\mathbb{Q}$ of SL_2; $\mu_{\mathrm{SL}_2(\mathbb{Q}_v),\omega}$ is the corresponding measure on $\mathrm{SL}_2(\mathbb{Q}_v)$ defined via ω, as described in previous section.

In the next two sections, we will show that

$$\mu_{\mathrm{SL}_2(\mathbb{Q}_p),\omega}(\mathrm{SL}_2(\mathbb{Z}_p)) = \frac{|\,\mathrm{SL}_2(\mathbb{F}_p)|}{p^3} = 1 - p^{-2}$$

and

$$\mu_{\mathrm{SL}_2(\mathbb{R}),\omega}(\mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R})) = \zeta(2) = \frac{\pi^2}{6},$$

obtaining Theorem 120

9.3. **Volume of $\mathrm{SL}_2(\mathbb{Z}_p)$.** In this section, we will use the Haar measure $\mu_{\mathrm{SL}_2(\mathbb{Q}_p),\omega}$ induced by the top form $\omega = \frac{1}{x} dx \wedge dy \wedge dz$ (as defined in § 7.4) to compute the volume of $\mathrm{SL}_2(\mathbb{Z}_p)$. Indeed, we have a surjective map $p : \mathrm{SL}_2(\mathbb{Z}_p) \to \mathrm{SL}_2(\mathbb{F}_p)$ with kernel

$$N = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, c \in 1 + p\mathbb{Z}_p; b, d \in p\mathbb{Z}_p \right\}.$$

The surjectivity of p is shown in the following lemma:

Lemma 125. *Let $N \in \mathbb{Z}_{>0}$. The group homomorphism $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is surjective.*

Proof. Indeed, we want to show that for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z})$ such that $ad - bc - Nm = 1$ for some $m \in \mathbb{Z}$ then there exists $B \in \mathrm{SL}_2(\mathbb{Z})$ such that $B \equiv A \pmod{N}$. From $ad - bc - Nm = 1$, we know $\gcd(d, c, N) = 1$ so there exists $n \in \mathbb{Z}$ such that $\gcd(c, d + Nn) = 1$ (for example, by Chinese Remainder Theorem, we can choose n such that $d + Nn \equiv 1 \pmod{p}$ for $p|c, p \nmid N$ and $d + Nn \equiv d \pmod{p}$ for $p \mid c, p \mid N$, i.e. $p \nmid d$). By replacing d with $d + Nn$, we can assume that $\gcd(d, c) = 1$. We want to find $B = \begin{pmatrix} a + Ne & b + Nf \\ c & d \end{pmatrix}$ such that $ad - bc + N(de - cf) = 1$, or $m = de - cf$. As $\gcd(c, d) = 1$, there exists $e, f \in \mathbb{Z}$ such that $m = de - cf$, as desired. ☐

Because $|\mathrm{SL}_2(\mathbb{F}_p)| = p(p^2 - 1)$ so by the left-invariance of the measure, we find

$$\mu_{\mathrm{SL}_2(\mathbb{Q}_p),\omega}(\mathrm{SL}_2(\mathbb{Z}_p)) = |\mathrm{SL}_2(\mathbb{F}_p)| \mu_{\mathrm{SL}_2(\mathbb{Q}_p),\omega}(N) = p(p^2 - 1)\mu_{\mathrm{SL}_2(\mathbb{Q}_p),\omega}(N).$$

We have

$$\begin{aligned} \mu_{\mathrm{SL}_2(\mathbb{Q}_p),\omega}(N) &= \int_N |a^{-1}|_p dadbdc = \int_N dadbdc, \\ &= \int_{a,c \in 1 + p\mathbb{Z}_p, b \in p\mathbb{Z}_p} dadbdc, \\ &= \int_{p\mathbb{Z}_p} \int_{p\mathbb{Z}_p} \int_{p\mathbb{Z}_p} dadbdc, \\ &= (\mu_p(p\mathbb{Z}_p))^3 = p^{-3}. \end{aligned}$$

Thus, $\mu_{\mathrm{SL}_2(\mathbb{Q}_p),\omega}(\mathrm{SL}_2(\mathbb{Z}_p)) = (1 - p^{-2})$.

9.4. **Volume of** $\mathrm{SL}_2(\mathbb{Z})\setminus\mathrm{SL}_2(\mathbb{R})$**.** In this section, we compute the volume of $\mathrm{SL}_2(\mathbb{Z})\setminus\mathrm{SL}_2(\mathbb{R})$ by determine its fundamental domain.

9.4.1. *Volume form of* $\mathrm{SL}_2(\mathbb{R})$. From § 7.4, we know that SL_2 over $\mathbb{Q}$ has a unique left-invariant volume form ω up to scalar over $\mathbb{Q}^\times$. In particular, over open set $\left\{\begin{pmatrix} x & y \\ z & t\end{pmatrix}\in\mathrm{SL}_2(\mathbb{R}) : x\neq 0\right\}$ of $\mathrm{SL}_2(\mathbb{R})$ then $\omega = x^{-1}dx\wedge dy\wedge dz$.

Over $\mathbb{R}$, every element in $\mathrm{SL}_2(\mathbb{R})$ is uniquely expressed as product

$$\begin{pmatrix}\cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}\alpha & 0 \\ 0 & \alpha^{-1}\end{pmatrix}\begin{pmatrix}1 & u \\ 0 & 1\end{pmatrix}$$

where $\varphi\in[0,2\pi),\alpha>0,u\in\mathbb{R}$. Hence, under change of coordinates $x=\alpha\cos\varphi, y=\alpha u\cos\varphi-\alpha^{-1}\sin\varphi$ and $z=\alpha\sin\varphi$, we find that $\omega_\mathbb{R}$ can be globally expressed as $\omega_\mathbb{R}=\alpha d\varphi\wedge d\alpha\wedge du$

9.4.2. *Fundamental domain of* $\mathrm{SL}_2(\mathbb{Z})\setminus\mathrm{SL}_2(\mathbb{R})$. First, we denote the upper half-plane $\mathcal{H}=\{z\in\mathbb{C}:\Im(z)>0\}$. For $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{R}), z\in\mathbb{C}$, we define $gz:=\frac{az+b}{cz+d}$.

Proposition 126. *We have a smooth action of* $\mathrm{SL}_2(\mathbb{R})$ *on* $\mathcal{H}$ *via*

$$\begin{aligned}\Phi:\mathrm{SL}_2(\mathbb{R})\times\mathcal{H}&\to\mathcal{H}\\ (g,z)&\mapsto gz.\end{aligned}$$

This action is transitive and the special orthogonal group

$$\mathrm{SO}_2(\mathbb{R})=\left\{\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}:\theta\in\mathbb{R}\right\}$$

is the stabiliser of i*, inducing a homeomorphism*

$$\phi:\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})\to\mathcal{H}$$

sending $g\mapsto gi$*. Furthermore,* $\mathrm{PSL}_2(\mathbb{R})=\mathrm{SL}_2(\mathbb{R})/\{\pm1\}$ *acts faithfully on* $\mathcal{H}$*.*

Proof. As $\Im(gz)=\frac{\Im(z)}{|cz+d|^2}>0$ so $gz\in\mathcal{H}$, meaning each $g\in\mathrm{SL}_2(\mathbb{R})$ induces a smooth map from $\mathcal{H}$ to $\mathcal{H}$ (called *linear fractional transformation*) with inverse g^{-1}. Furthermore, one can also check $g(g'z)=(gg')z$ so we have an action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{H}$.

Next, we show this action ϕ is smooth. We first choose a chart for $\mathrm{SL}_2(\mathbb{R})$. Without loss of generality, let $U_a=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{R}):a\neq 0\right\}$ be a open subset of $\mathrm{SL}_2(\mathbb{R})$ with chart $\phi:U_a\to V$ where $V=\{(a,b,c)\in\mathbb{R}^3:a\neq 0\}$ sending $\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ to (a,b,c). Under this chart, the map ϕ is

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}\times z\mapsto\frac{az+b}{cz+\frac{bc+1}{a}},$$

which is smooth on $U_a\times\mathcal{H}$ because it is composition of smooth maps.

This action is transitive as for any $z=x+iy$ then $y^{-1/2}\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}\in\mathrm{SL}_2(\mathbb{R})$ maps i to z. One can also check that

$$\mathrm{SO}_2(\mathbb{R})=\left\{\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}:\theta\in\mathbb{R}\right\}.$$

is the stabiliser of i. Overall, we have a smooth and transitive action of the Lie group $\mathrm{SL}_2(\mathbb{R})$ onto the smooth manifold $\mathcal{H}$, we obtain a diffeomorphism

$$\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})=G/\mathrm{Stab}(i)\to\mathcal{H},$$

sending $g \mapsto gi$. $\square$

Under the action of $\mathrm{PSL}_2(\mathbb{Z})$, $\mathcal{H}$ has fundamental domain:

$$D = \{z \in \mathcal{H} : |z| > 1, \mathrm{Re}(z) < 1/2\}.$$

The action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathcal{H}$ commutes with the left action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$. We then find

Proposition 127. *(a) The fundamental domain for the action of* $\mathrm{PSL}_2(\mathbb{Z})$ *on* $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ *is*

$$\phi^{-1}(D) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : |u| < 1/2, 0 < \alpha < \frac{1}{\sqrt{1-u^2}} \right\}$$

(b) The fundamental domain for the left-action of $\mathrm{SL}_2(\mathbb{Z})$ *on* $\mathrm{SL}_2(\mathbb{R})$ *is* $\phi^{-1}(D)K$ *where*

$$K = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & \varphi \\ 0 & 1 \end{pmatrix} : \varphi \in [0, 2\pi), \alpha > 0 \right\} \cong \mathrm{SO}_2(\mathbb{R})/\{\pm 1\}.$$

Proof. (a) Indeed, as ϕ is homeomorphism, $\phi^{-1}(D)$ is open connected. As no two points in D belong to the same $\mathrm{PSL}_2(\mathbb{Z})$-orbit, no two points in $\phi^{-1}(D)$ that belongs to the same $\mathrm{PSL}_2(\mathbb{Z})$-orbit. We also have $\phi^{-1}(\overline{D}) = \overline{\phi^{-1}(D)}$, hence, knowing $\mathcal{H} = \bigcup_{\gamma \in \mathrm{PSL}_2(\mathbb{Z})} \gamma \overline{D}$ implies $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) = \bigcup_{\gamma \in \mathrm{PSL}_2(\mathbb{Z})} \gamma \overline{\phi^{-1}(D)}$. We are done.

(b) From (a), we have

$$\begin{aligned} \mathrm{SL}_2(\mathbb{R}) &= \bigcup_{\gamma \in \mathrm{PSL}_2(\mathbb{Z})} \gamma \overline{\phi^{-1}(D)}\, \mathrm{SO}_2(\mathbb{R}), \\ &= \bigcup_{\gamma \in \mathrm{SL}_2(\mathbb{Z})} \gamma \overline{\phi^{-1}(D)}\, \mathrm{SO}_2(\mathbb{R})/\{\pm 1\}, \\ &= \bigcup_{\gamma \in \mathrm{SL}_2(\mathbb{Z})} \gamma \overline{\phi^{-1}(D)} K. \end{aligned}$$

Also from (a), no two points in $\phi^{-1}(D)K$ belong to the same $\mathrm{SL}_2(\mathbb{Z})$-orbit, else we can find two points in $\phi^{-1}(D)$ belong to the same $\mathrm{PSL}_2(\mathbb{Z})$-orbit. $\square$

Thus, from this proposition, we find

$$\begin{aligned} \mu_{\mathrm{SL}_2(\mathbb{R}),\omega}(\mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R})) &= \int_{\phi^{-1}(D)K} \alpha d\alpha du d\varphi, \\ &= \int_{u=-1/2}^{1/2} \int_{\alpha=0}^{(1-u^2)^{-1/2}} \int_{\varphi=0}^{\pi} \alpha d\varphi d\alpha du, \\ &= \pi^2/6. \end{aligned}$$

10. The Tamagawa number of special linear groups

In this section, we will prove the following theorem

Theorem 128. $\tau_{\mathbb{Q}}(\mathrm{SL}_n) = 1$.

We will induct on n. The case $n = 2$ was proved in the previous section.

10.1. **The action of SL_n on $\mathbb{G}_a^n$.** We study the action of SL_n on $V = \mathbb{G}_a^n$ by right multiplication. In particular, we determine the orbits and stabilisers of this action and describe measures on these spaces.

One can prove the following

Proposition 129. *If k is a division algebra, the orbits of $\mathrm{SL}_n(k)$ acting on k^n are $\{0\}$ and $k^n - \{0\} = e_1 \mathrm{SL}_n(k)$, where $e_1 = (1, 0, \ldots, 0) \in k^n$.*

Let SL_{n,e_1} be the stabiliser of e_1 under this action. Then SL_{n,e_1} is the semidirect product $\mathrm{SL}_{n-1} \ltimes \mathbb{G}_a^{n-1}$. In particular, every element in $\mathrm{SL}_{n,e_1}(k)$ has the form

$$\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}$$

where $c^t \in k^{n-1}, d \in \mathrm{SL}_{n-1}(k)$.

10.1.1. *Measures on orbits and stabilisers.* The map $\mathrm{SL}_n(k) \to e_1 \mathrm{SL}_n(k)$ sending $g \mapsto e_1 g$ gives rise to a bijection $\mathrm{SL}_{n,e_1}(k)\backslash \mathrm{SL}_n(k) \xrightarrow{\sim} e_1 \cdot \mathrm{SL}_n(k)$. When $k = \mathbb{Q}_v$ or $k = \mathbb{A}$, this induces a homeomorphism with respect to the topology of k. Because the map $g \mapsto e_1 g$ is also SL_n-invariant, the SL_n-invariant volume form $dv = dv_1 \wedge \cdots \wedge dv_n$ on $e_1 \cdot \mathrm{SL}_n(k)$ (it is invariant because $\det(g) = 1$ for $g \in \mathrm{SL}_n(k)$) induces an algebraic differential form θ on SL_n, satisfying $dh \wedge \theta = dg$, where dh, dg are invariant volume forms on SL_{n,e_1} and SL_n, respectively. This implies that for any continuous function f on $\mathrm{SL}_n(\mathbb{A})$ with compact support, we find

$$\int_{\mathrm{SL}_n(\mathbb{A})} f(g)|dg|_{\mathbb{A}} = \int_{e_1\cdot \mathrm{SL}_n(\mathbb{A})} \left(\int_{\mathrm{SL}_{n,e_1}(\mathbb{A})} f(hg)|dh|_{\mathbb{A}} \right) |d(e_1 \cdot g)|_{\mathbb{A}} \tag{6}$$

By the uniqueness of the $\mathrm{SL}_n(\mathbb{A})$-invariant measure on the quotient space $\mathrm{SL}_{n,e_1}(\mathbb{A})\backslash \mathrm{SL}_n(\mathbb{A})$ induced from $|dh|_{\mathbb{A}}$ and $|dg|_{\mathbb{A}}$, from (6), we find

$$\int_{e_1\cdot \mathrm{SL}_n(\mathbb{A})} f(v)|dv|_{\mathbb{A}} = \int_{\mathrm{SL}_{n,e_1}(\mathbb{A})\backslash \mathrm{SL}_n(\mathbb{A})} f(e_1 g)|dg|_{\mathbb{A}},$$

where f is a continuous function on $\mathbb{A}^n$ with compact support and $|dg|_{\mathbb{A}}$ is the $\mathrm{SL}_n(\mathbb{A})$-invariant measure on $\mathrm{SL}_{n,e_1}(\mathbb{A}) \setminus \mathrm{SL}_n(\mathbb{A})$.

On the other hand, we have the following proposition

Proposition 130. $\mathbb{A}^n - e_1 \mathrm{SL}_n(\mathbb{A})$ *has measure* 0.

Proof. Note that if $v = (v_1, \ldots, v_n) \in \mathbb{A}$ such that $v_i \in \mathbb{A}^\times$ for some $1 \le i \le n$ then $v \in e_1 \mathrm{SL}_n(\mathbb{A})$. Therefore, $\mathbb{A}^n - e_1 \mathrm{SL}_n(\mathbb{A}) \subset (\mathbb{A} - \mathbb{A}^\times)^n$. Furthermore, $\mathbb{A} - \mathbb{A}^\times$ lies in the union of open sets of the form $U_S \times \prod_{p \notin S} p^{k_p}\mathbb{Z}_p$, where S is a finite set of places of $\mathbb{Q}$ containing the infinite place, U_S is open in $\prod_{v\in S} \mathbb{Q}_v$, $k_p \ge 1$ for all $p \in S$. We also know that $U_S \times \prod_{p \notin S} p^{k_p}\mathbb{Z}_p$ has measure 0. Hence, $\mathbb{A} - \mathbb{A}^\times$ is a null set (see § 3), meaning it has measure 0 (after enlarging our measure space on $\mathbb{A}$ to include give null sets as having measure 0). ∎

Thus, we can rewrite the above identity as

$$\int_{\mathbb{A}^n} f(v)|dv|_{\mathbb{A}} = \int_{\mathrm{SL}_{n,e_1}(\mathbb{A})\backslash \mathrm{SL}_n(\mathbb{A})} f(e_1 g)|dg|_{\mathbb{A}}. \tag{7}$$

10.2. **Using Poisson summation formula.** For $f \in \mathcal{S}(\mathbb{A}^n)$, we let

$$I(f) := \int_{\mathrm{SL}_n(\mathbb{Q})\backslash \mathrm{SL}_n(\mathbb{A})} \left(\sum_{x\in\mathbb{Q}^n} f(xg) \right) |dg|_{\mathbb{A}}.$$

Assuming for a moment the convergence of this integral for any $f \in \mathcal{S}(\mathbb{A}^n)$, we will prove the following

Proposition 131. *For any $f \in \mathcal{S}(\mathbb{A}^n)$, we have $I(f) = I(\widehat{f})$.*

Proof. For $g \in \mathrm{GL}_n(\mathbb{A})$, let $f_g(x) := f(xg)$, we find

$$\begin{aligned}\widehat{f_g}(x) &= \int_{\mathbb{A}^n} f(yg)\exp_{\mathbb{A}}(yx^t)|dy|_{\mathbb{A}},\\ &= |\det(g)|_{\mathbb{A}}^{-1}\int_{\mathbb{A}^n} f(y)\exp_{\mathbb{A}}(yg^{-1}x^t)|dy|_{\mathbb{A}},\\ &= |\det(g)|_{\mathbb{A}}^{-1}\int_{\mathbb{A}^n} f(y)\exp_{\mathbb{A}}(y(xg^{-t})^t)|dy|_{\mathbb{A}}\\ &= |\det(g)|_{\mathbb{A}}^{-1}\hat{f}(xg^{-t}).\end{aligned}$$

By applying the Poisson summation formula for f_g, we find

$$\sum_{x\in\mathbb{Q}^n} f(xg) = |\det(g)|_{\mathbb{A}}^{-1}\sum_{x\in\mathbb{Q}^n}\widehat{f}(xg^{-t}).$$

Let $g \in \mathrm{SL}_n(\mathbb{Q})$ and noting that $g \mapsto g^{-t}$ is a measure-preserving homeomorphism on $\mathrm{SL}_n(\mathbb{A})$, we find $I(f) = I(\widehat{f})$. □

10.3. **Write the integral over** $\mathrm{SL}_n(\mathbb{Q})\backslash \mathrm{SL}_n(\mathbb{A})$ **into orbits.** Using the results in § 10.1, we have

$$\begin{aligned}I(f) &:= \int_{\mathrm{SL}_n(\mathbb{Q})\backslash\mathrm{SL}_n(\mathbb{A})} F_f(g)|dg|_{\mathbb{A}},\\ &= \int_{\mathrm{SL}_n(\mathbb{Q})\backslash\mathrm{SL}_n(\mathbb{A})}\left(f(0) + \sum_{x\in e_1\cdot\mathrm{SL}_n(\mathbb{Q})} f(xg)\right)|dg|_{\mathbb{A}},\\ &= f(0)\tau_{\mathbb{Q}}(\mathrm{SL}_n) + \int_{\mathrm{SL}_n(\mathbb{Q})\backslash\mathrm{SL}_n(\mathbb{A})}\sum_{\gamma\in\mathrm{SL}_{n,e_1}(\mathbb{Q})\backslash\mathrm{SL}_n(\mathbb{Q})} f(e_1\gamma g)|dg_{\mathbb{A}},\\ &= f(0)\tau_{\mathbb{Q}}(\mathrm{SL}_n) + \int_{\mathrm{SL}_{n,e_1}(\mathbb{Q})\backslash\mathrm{SL}_n(\mathbb{A})} f(e_1 g)|dg|_{\mathbb{A}},\\ &= f(0)\tau_{\mathbb{Q}}(\mathrm{SL}_n) + \tau_{\mathbb{Q}}(\mathrm{SL}_{n,e_1})\int_{\mathrm{SL}_{n,e_1}(\mathbb{A})\backslash\mathrm{SL}_n(\mathbb{A})} f(e_1 g)|dg|_{\mathbb{A}},\\ &= f(0)\tau_{\mathbb{Q}}(\mathrm{SL}_n) + \tau_{\mathbb{Q}}(\mathrm{SL}_{n,e_1})\int_{\mathbb{A}^n} f(v)|dv|_{\mathbb{A}},\\ &= f(0)\tau_{\mathbb{Q}}(\mathrm{SL}_n) + \tau_{\mathbb{Q}}(\mathrm{SL}_{n,e_1})\widehat{f}(0),\\ &= f(0)\tau_{\mathbb{Q}}(\mathrm{SL}_n) + \widehat{f}(0).\end{aligned}$$

Replacing f by $\widehat{f}$ in the above and noting that $f(0) = \hat{\hat{f}}(0)$ by Fourier inversion formula, we find

$$I(\widehat{f}) = \widehat{f}(0)\tau_{\mathbb{Q}}(\mathrm{SL}_n) + f(0).$$

From Proposition 131, we find

$$(\tau_{\mathbb{Q}}(\mathrm{SL}_n) - 1)(f(0) - \widehat{f}(0)) = 0.$$

There exists $f \in \mathcal{S}(\mathbb{A}^n)$ so $f(0) \neq \widehat{f}(0)$. Thus, we find $\tau_{\mathbb{Q}}(\mathrm{SL}_n) = 1$.

11. The Tamagawa number of symplectic groups

We first recall the definition of Sp_{2n}. Let $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ be a $2n$-by-$2n$ matrix. The symplectic group is defined as

$$\mathrm{Sp}_{2n}(k) := \{M \in M_{2n\times 2n}(k) : M^t J_n M = J_n\}.$$

In this section, we will focus on proving the following theorem

Theorem 132. *For all $n \geq 1$ then $\tau_{\mathbb{Q}}(\mathrm{Sp}_{2n}) = 1$.*

We will prove this theorem by induction on n. When $n = 1$ then $\mathrm{Sp}_2 = \mathrm{SL}_2$, hence $\tau_{\mathbb{Q}}(\mathrm{Sp}_2) = \tau_{\mathbb{Q}}(\mathrm{SL}_2) = 1$.

11.1. The action of Sp_{2n} on $\mathbb{G}_a^{2n}$.

We study the action of Sp_{2n} on $\mathbb{G}_a^{2n}$ by right multiplication.

Proposition 133. *If k is a division algebra then the orbits of $\mathrm{Sp}_{2n}(k)$ acting on k^{2n} is $\{0\}$ and $k^{2n} - \{0\} = e_1 \mathrm{Sp}_{2n}(k)$, where $e_1 = (1, 0, \ldots, 0)$.*

Let Sp_{2n,e_1} be the stabiliser of e_1, then Sp_{2n,e_1} is isomorphic to the semidirect product $\mathrm{Sp}_{2n-2} \ltimes (\mathbb{G}_a^{2n-2} \ltimes \mathbb{G}_a)$.

Proof. Let $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $A, B, C, D \in M_{n\times n}$. For X to stabilise e means $A = \begin{pmatrix} 1 & 0 \\ * & * \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix}$.

Furthermore, $X \in \mathrm{Sp}_{2n}$ if and only if $A^tC - C^tA = B^tD - D^tB = 0$ and $A^tD - C^tB = I_n$. Thus, $X \in \mathrm{Sp}_{2n,e_1}$ if and only if $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_1 & a & 0 & c \\ x & y_1 & 1 & y_2 \\ x_2 & b & 0 & d \end{pmatrix}$ where $\begin{pmatrix} y_1 & y_2 \end{pmatrix} = \begin{pmatrix} x_2^t & -x_1 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{Sp}_{2n-2}$. Here x_i are $(n-1)\times 1$ vectors, y_i are $1 \times (n-1)$ vectors. □

11.2. Computing the Tamagawa number of Sp_{2n}.

With the same argument as in the case of SL_n, one can show that for $f \in \mathcal{S}(\mathbb{A}^{2n})$, we find

$$\int_{\mathbb{A}^{2n}} f(v)|dv|_{\mathbb{A}} = \int_{\mathrm{Sp}_{2n,e_1}(\mathbb{A})\backslash \mathrm{Sp}_{2n}(\mathbb{A})} f(e_1 g)|dg|_{\mathbb{A}}.$$

Define

$$I(f) := \int_{\mathrm{Sp}_{2n}(\mathbb{Q})\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \left(\sum_{x\in\mathbb{Q}^n} f(xg) \right) |dg|_{\mathbb{A}}.$$

By applying Poisson summation formula as in the case of SL_n, we find $I(f) = I(\widehat{f})$.

Finally, similarly to the case of SL_n, we have

$$I(f) = f(0)\tau_{\mathbb{Q}}(\mathrm{Sp}_{2n}) + \tau_{\mathbb{Q}}(\mathrm{Sp}_{2n,e_1})\widehat{f}(0) = f(0)\tau_{\mathbb{Q}}(\mathrm{Sp}_{2n}) + \widehat{f}(0).$$

Replacing f by $\widehat{f}$ in the above equation and noting that $\hat{\hat{f}}(0) = f(0)$ by Fourier inversion formula, we find

$$I(\hat{f}) = \hat{f}(0)\tau_{\mathbb{Q}}(\mathrm{Sp}_{2n}) + f(0).$$

Thus, we obtain $\tau_{\mathbb{Q}}(\mathrm{Sp}_{2n}) = 1$.

12. The Tamagawa number of special orthogonal groups

In this section, we will show that the Tamagawa number for the special orthogonal group $\mathrm{SO}_{q,\mathbb{Q}}$ (with respect to a non-degenerate quadratic form q) is 2 (see Theorem 134). In this section, we follow [Mar66], [Igu78, Chapter 4] and [Hid20].

12.1. **Orthogonal groups.**

Let V be a vector space over $\mathbb{Q}$ of dimension $n \geq 3$. A map $q : V \to \mathbb{Q}$ is called a *quadratic form on V over* $\mathbb{Q}$ if it satisfies the following conditions:

(1) The function $b : V \times V \to \mathbb{Q}$, defined by
$$b(x, y) := q(x + y) - q(x) - q(y),$$
is a symmetric bilinear form.

(2) For $\lambda \in \mathbb{Q}$ and $x \in V$, we have $q(\lambda x) = \lambda^2 q(x)$.

Throughout this section, we will always assume that q is nondegenerate, i.e. b is nondegenerate.

A morphism between two quadratic forms q, q' is a linear map $f : V \to V$ such that $q' \circ f = q$. The automorphism group of a quadratic form q over $\mathbb{Q}$ is denoted as $\mathrm{O}_q(\mathbb{Q})$, called the *orthogonal group* of (V, q).

For any $\mathbb{Q}$-algebra R, q induces a quadratic form $q_R : V \otimes_{\mathbb{Q}} R \to R$ over R by extension of scalars. Its automorphism group is denoted by $\mathrm{O}_q(R)$. Thus, we have defined an algebraic group O_q over $\mathbb{Q}$ corresponding to the quadratic form $q : V \to \mathbb{Q}$.

Furthermore, O_q is an affine algebraic group. Indeed, we fix a choice of basis $\{e_1, \dots, e_n\}$ for V, a quadratic form q on V then corresponds to a symmetric matrix B_q defined by $(B_q)_{ij} := \frac{1}{2}(q(e_i + e_j) - q(e_i) - q(e_j))$. One can show that two quadratic forms q, q'' on V are isomorphic over $\mathbb{Q}$ if $B_{q'} = T^t B_q T$ for some $T \in \mathrm{GL}_n(\mathbb{Q})$. Thus, for any $\mathbb{Q}$-algebra R, we can describe $\mathrm{O}_q(R)$ as
$$\mathrm{O}_q(R) = \{x \in \mathrm{GL}_n(R) : B_q = x^t B_q x\}.$$
Let SO_q be the closed algebraic subgroup of O_q that consists of automorphisms of q having determinant 1.

Our main result in this section is the following

Theorem 134. $\tau_{\mathbb{Q}}(\mathrm{SO}_q) = 2$.

To prove this, we induct on $n = \dim V$. We will take for granted that $\tau_{\mathbb{Q}}(\mathrm{SO}_q) = 2$ for $n = 3, 4$. The proof for these cases can be found at [Wei82, p. 65, Theorem 3.7.1]. To use induction, we first start by studying the action of SO_q on $V = \mathbb{G}_a^n$.

12.2. **Orbits and stabilisers of the action of SO_q on $\mathbb{G}_a^n$.**

We study the action of SO_q on $V = \mathbb{G}_a^n$ by right multiplication.

12.2.1. *Orbits.*

The orbits under the action of SO_q on V are described in the below proposition.

Proposition 135. *Let k be an extension of $\mathbb{Q}$. The orbits under the action of* $\mathrm{SO}(k)$ *on $V(k)$ by right-multiplication are $U(i)_k := q_k^{-1}(i) - \{0\}$ for $i \in k$ and $\{0\}$ (note $U(i)_k$ can be empty for some i).*

This follows from Witt's theorem.

Lemma 136 (Witt's theorem)**.** *Let k be an extension of $\mathbb{Q}$, then two nonzero points $x, y \in V(k) = k^n$ belong to the same orbit under the action of* $\mathrm{SO}_q(k)$ *by right-multiplication if and only if $q_k(x) = q_k(y)$.*

12.2.2. *Stabilisers.* For $0 \neq v \in V(\mathbb{Q})$, let $\mathrm{SO}_{v,q}$ be the stabiliser of v under the action of SO_q on V, i.e. for a $\mathbb{Q}$-algebra R, we denote

$$\mathrm{SO}_{v,q}(R) = \{g \in \mathrm{O}_q(R) : vg = v\}.$$

We will show that $\mathrm{SO}_{v,q}$ is a linear algebraic group by the following proposition

Proposition 137. *If $q(v) \neq 0$ then $\mathrm{SO}_{v,q}$ is isomorphic to a special orthogonal group of dimension $n-1$. If $q(v) = 0$ then $\mathrm{SO}_{v,q}$ is isomorphic to the semidirect product of $\mathbb{G}_a^{n-2}$ acting on a special orthogonal group of dimension $n-2$.*

Proof. We will describe the structure of $\mathrm{SO}_{v,q}(R)$ in the two cases where $q(v) = 0$ and $q(v) \neq 0$. For convenience, we will restrict the discussion to $\mathrm{SO}_{v,q}(\mathbb{Q})$, as the case $\mathrm{SO}_{v,q}(R)$ for any $\mathbb{Q}$-algebra R is completely similar.

If $q(v) \neq 0$ then we will show SO_v is a special orthogonal group of dimension $n-1$. Let $W_v = (\mathbb{Q}v)^\perp := \{v' \in V : q(v, v') = 0\}$ (for convenience, we refer to $b(v, v')$ as $q(v, v')$) then W_v is a $\mathbb{Q}$-vector space of dimension $n-1$. Indeed, we know $q(v) \neq 0$ so $q(v, v) \neq 0$. Hence, for any basis $v, v_1, \ldots, v_{n-1}$ of V, one can choose $c_i \in \mathbb{Q}$ so that $q(v, v_i + c_i v) = 0$. This means $W_v = \mathrm{span}_{\mathbb{Q}}\{v_1 + c_1 v, \ldots, v_{n-1} + c_{n-1} v\}$.

We note that if $g \in \mathrm{SO}_v$ then g preserves W_v, as $0 = q(v, v') = q(gv, gv') = q(v, gv')$ for $v' \in W_v$. This means $\mathrm{SO}_v \subset \mathrm{SO}_{q|_{W_v}}$. Conversely, given $g \in \mathrm{SO}_{q|_{W_v}}$ then we can extend g to action on V by letting $gv = v$, as $v \notin W_v$. Thus, $\mathrm{SO}_v = \mathrm{SO}_{q|_{W_v}}$, i.e. SO_v is the orthogonal group corresponding to the quadratic form $q|_{W_v}$.

If $q(v) = 0$ but $v \neq 0$, the restriction of q to $\mathbb{Q}v$ is trivial. Since q is nondegenerate, there exists $v' \in V$ independent from v such that $q(v, v') = 1$. Then $q(v' - xv) = \frac{1}{2}q(v' - xv, v' - xv) = q(v') - x$ for $x \in \mathbb{Q}$. Thus, by taking $x = q(v')$ and replacing v' by $v' - xv$, we may assume that $q(v') = 0$. It follows $W = (\mathbb{Q}v \oplus \mathbb{Q}v')^\perp$ has dimension $n-2$, as for any $w \in W$, we can find $c, c' \in \mathbb{Q}$ so $q(w + cv + c'v', v) = q(w + cv + c'v', v') = 0$. Thus, under a choice of basis $v, v_1, \ldots, v_{n-2}, v'$ of V where $v_1, \ldots, v_{n-2}$ is a basis of W, q has the matrix form

$$B_q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & S & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

where S is a $(n-2) \times (n-2)$ symmetric matrix corresponding to $q|_W$. From this, one can show that $\mathrm{SO}_{v,q}$ is the semidirect product $\mathrm{SO}_{q|_W} \ltimes \mathbb{G}_a^{n-2}$, i.e. an element in $\mathrm{SO}_{v,q}(\mathbb{Q})$ has the form

$$\begin{pmatrix} 1 & 0 & 0 \\ w & A & 0 \\ -\frac{1}{2}w^t S w & -w^t S & 1 \end{pmatrix}$$

where $w \in \mathbb{G}_a^{n-2}$ and $A \in \mathrm{SO}_{q|_W}$. □

12.2.3. *Measures on orbits and stabilisers.* In this section, we will define a SO_q-invariant volume form on the orbits $U(i)$. This volume form allows us to connects the Tamagawa measure on SO_q (see (9)) with the Tamagawa measure on the affine space V (see (8)).

Let $dv = dv_1 \wedge \cdots \wedge dv_n$ be the volume form on V. As q is nondegenerate, we find $q^* dx = d(q(v))$ is a nowhere vanishing 1-form on $V(\mathbb{Q}) - \{0\}$ [15]. Because we have a submersion $q : V - \{0\} \to \mathbb{G}_a$

[15] We know $d(q(v)) = \sum_{k=1}^n \frac{\partial q}{\partial v_i} dv_i$. It suffices to prove that if for $x \in V(\mathbb{Q})$ so that $\frac{\partial q}{\partial v_i}(x) = 0$ for all $1 \leq i \leq n$, then $x = 0$. Indeed, because q is nondegenerate, there exists a non-singular matrix A such that $q(v) = v^T A v = v_1(Av)_1 + \ldots + v_n(Av)_n$. We find $\frac{\partial q}{\partial v_i} = (Av)_i + (v^T A)_i$. Therefore, if there is $x \in V(\mathbb{Q})$ so that $\frac{\partial q}{\partial v_i}(x) = 0$ for all $1 \leq i \leq n$ then $x^T A = -(Ax)^T = -x^T A^T$ or $2x^T A = 0$ since A is symmetric. This implies $x = 0$.

so there exists an algebraic nowhere-vanishing differential form θ of degree $n-1$ on $V(\mathbb{Q})-\{0\}$ such that

$$\theta \wedge d(q(v)) = dv.$$

Observe that θ is SO_q-invariant since $q(vg)=q(v)$ and $d(vg)=\det(g)dv=dv$.

By restricting θ to $U(i)\subset V-\{0\}$, we obtain a G-invariant volume form θ_i on $U(i)$ over $\mathbb{Q}$. For a local field $\mathbb{Q}_v$ of $\mathbb{Q}$, θ_i defines G-invariant volume forms $|\theta_i|_v$ on $U(i)_{\mathbb{Q}_v}$ for any $i\in\mathbb{Q}_v$, satisfying the integration along fibers of the submersion $q_{\mathbb{Q}_v}$

$$\int_{V(\mathbb{Q}_v)-\{0\}} f|dx|_v = \int_{\mathbb{Q}_v}\left(\int_{U(i)_{\mathbb{Q}_v}} f|\theta_i|_v\right)|di|_v.$$

Over $\mathbb{A}_{\mathbb{Q}}$, we also have a similar result

$$\int_{V(\mathbb{A})} f|dx|_{\mathbb{A}} = \int_{\mathbb{A}}\left(\int_{U(i)_{\mathbb{A}}} f|\theta_i|_{\mathbb{A}}\right)|di|_{\mathbb{A}}. \tag{8}$$

On the other hand, for $i\in\mathbb{Q}$ and if $0\neq v_i\in U(i)_k$ then by Witt's theorem, for any extension k of $\mathbb{Q}$, the map $\mathrm{SO}_q(k)\to U(i)_k$ sending $g\mapsto v_ig$ give rise to a bijection $\mathrm{SO}_{q,v_i}(k)\setminus \mathrm{SO}_q(k)\xrightarrow{\sim} U(i)_k$. When $k=\mathbb{Q}_p,\mathbb{R}$ or $\mathbb{A}$, this induces a homeomorphism with respect to the topology of k. Because the map $g\mapsto v_ig$ is SO_q-invariant, the algebraic differential form θ_i on $U(i)$ induces an algebraic differential form θ_i' on SO_q, satisfying $dh\wedge\theta_i'=dg$, where dh,dg are invariant volume forms on SO_{q,v_i} and SO_q, respectively. Thus, for any continuous function F on $\mathrm{SO}_q(\mathbb{A})$ with compact support, we find

$$\int_{\mathrm{SO}_q(\mathbb{A})} F(g)|dg|_{\mathbb{A}} = \int_{U(i)_{\mathbb{A}}}\left(\int_{\mathrm{SO}_{q,v_i}(\mathbb{A})} F(hg)|dh|_{\mathbb{A}}\right)|\theta_i(v_i\cdot g)|_{\mathbb{A}}. \tag{9}$$

Here, $|dg|_{\mathbb{A}}$ and $|dh|_{\mathbb{A}}$ are the Tamagawa measures on $\mathrm{SO}_q(\mathbb{A})$ and $\mathrm{SO}_{q,v_i}(\mathbb{A})$, respectively.

12.2.4. *Measure on the orbit of* 0. For the purpose of the later section, we need a more detail study of the measure $|\theta_0|_k$ on $U(0)_k$, where k is either a local field $k=\mathbb{Q}_v$ or $k=\mathbb{A}_{\mathbb{Q}}$.

Recall from the proof of Proposition 137, if there is $0\neq v\in U(0)_k$, there exists $v'\in V$ so $q(v')=0$ and $q(v,v')=1$. We then let $W=(kv\oplus kv')^{\perp}$. One can then show that any $v''\in U(0)$, i.e. $0\neq v''$ and $q(v'')=0$, is either in $kv\oplus q|_W^{-1}(0)\cup kv'\oplus q|_W^{-1}(0)$ or can be written as

$$v''=-(2a)^{-1}q(w)v+w+av'$$

for $a\in k^{\times}$ and for any $w\in W-(q|_W)^{-1}(0)$. Because, $kv\oplus q|_W^{-1}(0)\cup kv'\oplus q|_W^{-1}(0)$ is a proper Zariski closed subset of $U(0)_k$ (for example, $kv\oplus q|_W^{-1}(0)$ contains all $x\in U(0)_k$ such that $q(v,x)=0$, and the polynomial $q(v,\cdot)$ does not vanish on $U(0)_k$), $|\theta_0|_k$ has measure 0 on this set. Hence, the map

$$\begin{aligned} i:\left(W-q|_W^{-1}(0)\right)\times\mathbb{G}_m &\to U(0)_k,\\ (w,a)&\mapsto -(2a)^{-1}q(w)v+w+av' \end{aligned}$$

has image being a Zariski dense open subset of $U(0)_k$, and $|\theta_0|_k$ is determined by its restriction to its open set $\left(W-q|_W^{-1}(0)\right)\times\mathbb{G}_m$. Because i is translation W-invariant and $\mathbb{G}_m$-invariant, $i^*\theta_0$, as an algebraic differential from, must also be invariant under these actions. Hence, $i^*\theta_0$ is $dx_1\wedge\cdots\wedge dx_{n-2}\wedge\frac{da}{a}$ up to a constant multiple of $\mathbb{Q}^{\times}$, where $dx_1\wedge\cdots\wedge dx_{n-2}$ is a W-invariant volume form on W and $\frac{da}{a}$ is a $\mathbb{G}_m$-invariant volume form on $\mathbb{G}_m$. This follows for a fixed $a\in k^{\times}$, we find

$$|\theta_0(au)|=|a|_k^{n-2}|\theta_0(u)|$$

for $u\in U(0)_k$.

12.3. **Writing the integral over $\mathrm{SO}_q(\mathbb{Q})\backslash\mathrm{SO}_q(\mathbb{A})$ into orbits.** To calculate $\tau_{\mathbb{Q}}(\mathrm{SO}_q)=\int_{\mathrm{SO}_q(\mathbb{Q})\backslash\mathrm{SO}_q(\mathbb{A})}|dg|_{\mathbb{A}}$ where $|dg|_{\mathbb{A}}$ is the Tamagawa measure on $\mathrm{SO}_q(\mathbb{A})$, we consider a general integral

$$\int_{\mathrm{SO}_q(\mathbb{Q})\backslash\mathrm{SO}_q(\mathbb{A})}F(g)|dg|_{\mathbb{A}}$$

for some left $\mathrm{SO}_q(\mathbb{Q})$-invariant continuous function F on $\mathrm{SO}_q(\mathbb{A})$ with compact support. We let $f:V(\mathbb{A})\to\mathbb{C}$ be a Schwartz-Bruhat function and let $F_f:\mathrm{SO}_q(\mathbb{A})\to\mathbb{C}$ be defined by

$$F_f(g):=\sum_{x\in V(\mathbb{Q})}f(xg).$$

Note that F_f is left $\mathrm{SO}_q(\mathbb{Q})$-invariant, and we temporarily assume the convergence of the below integral to find

$$I(f):=\int_{\mathrm{SO}_q(\mathbb{Q})\backslash\mathrm{SO}_q(\mathbb{A})}F_f(g)|dg|_{\mathbb{A}}, \tag{10}$$

$$=\int_{\mathrm{SO}_q(\mathbb{Q})\backslash\mathrm{SO}_q(\mathbb{A})}\left(f(0)+\sum_{\substack{i\in\mathbb{Q}\\ U(i)_{\mathbb{Q}}\neq\emptyset}}\sum_{x\in U(i)_{\mathbb{Q}}}f(xg)\right)|dg|_{\mathbb{A}}, \tag{11}$$

$$=f(0)\tau_{\mathbb{Q}}(\mathrm{SO}_q)+\int_{\mathrm{SO}_q(\mathbb{Q})\backslash\mathrm{SO}_q(\mathbb{A})}\sum_{\substack{i\in\mathbb{Q}\\ U(i)_{\mathbb{Q}}\neq\emptyset}}\sum_{\gamma\in\mathrm{SO}_{v_i}(\mathbb{Q})\backslash\mathrm{SO}_q(\mathbb{Q})}f(v_i\gamma g)|dg|_{\mathbb{A}}, \tag{12}$$

$$=f(0)\tau_{\mathbb{Q}}(\mathrm{SO}_q)+\sum_{\substack{i\in\mathbb{Q}\\ U(i)_{\mathbb{Q}}\neq\emptyset}}\int_{\mathrm{SO}_{v_i}(\mathbb{Q})\backslash\mathrm{SO}_q(\mathbb{A})}f(v_ig)|dg|_{\mathbb{A}}, \tag{13}$$

$$=f(0)\tau_{\mathbb{Q}}(\mathrm{SO}_q)+\sum_{\substack{i\in\mathbb{Q}\\ U(i)_{\mathbb{Q}}\neq\emptyset}}\int_{\mathrm{SO}_{v_i}(\mathbb{Q})\backslash\mathrm{SO}_{v_i}(\mathbb{A})}\left(\int_{U(i)_{\mathbb{A}}}f(v_ihg)|\theta(v_i\cdot g)|_{\mathbb{A}}\right)|dh|_{\mathbb{A}}, \tag{14}$$

$$=f(0)\tau_{\mathbb{Q}}(\mathrm{SO}_q)+\sum_{\substack{i\in\mathbb{Q}\\ U(i)_{\mathbb{Q}}\neq\emptyset}}\tau_{\mathbb{Q}}(\mathrm{SO}_{v_i})\int_{U(i)_{\mathbb{A}}}f(v)|\theta_i|_{\mathbb{A}}, \tag{15}$$

$$=f(0)\tau_{\mathbb{Q}}(\mathrm{SO}_q)+2\sum_{\substack{i\in\mathbb{Q}\\ U(i)_{\mathbb{Q}}\neq\emptyset}}\int_{U(i)_{\mathbb{A}}}f(v)|\theta_i|_{\mathbb{A}}, \tag{16}$$

$$=f(0)\tau_{\mathbb{Q}}(\mathrm{SO}_q)+2\sum_{i\in\mathbb{Q}}\int_{U(i)_{\mathbb{A}}}f(v)|\theta_i|_{\mathbb{A}}. \tag{17}$$

Some explanation for the above manipulations:

(11) Follows from knowing the orbits under the action of $\mathrm{SO}_q(\mathbb{Q})$ on $V(\mathbb{Q})$ (see Proposition 135).
(12) From § 12.2.3 where we have a bijection $U(i)_{\mathbb{Q}}\cong\mathrm{SO}_{v_i}(\mathbb{Q})\setminus\mathrm{SO}_q(\mathbb{Q})$.
(13) Because $\mathrm{SO}_q(\mathbb{Q})$ is discrete in $\mathrm{SO}_q(\mathbb{A})$, one can take the integral over $\mathrm{SO}_{v_i}(\mathbb{Q})\setminus\mathrm{SO}_q(\mathbb{A})$.
(14) Follows from (9) in § 12.2.3.
(15) Because $f(v_ig)$ is left $\mathrm{SO}_{v_i}(\mathbb{A})$-invariant.
(16) From the description of SO_{v_i} in Proposition 137, by inductive hypothesis, we know $\tau_{\mathbb{Q}}(\mathrm{SO}_{v_i})=2$ for all $i\in\mathbb{Q}$ and all $0\neq v_i\in U(i)_{\mathbb{Q}}$.

(17) By the Hasse principle, if $U(i)_{\mathbb{Q}} = \emptyset$ then $U(i)_{\mathbb{A}} = \emptyset$. Hence, we can take the sum over all $i \in \mathbb{Q}$.

We know that $\tau_{\mathbb{Q}}(\mathrm{SO}_q)$ is finite and we can also show that

$$\sum_{i \in \mathbb{Q}} \int_{U(i)_{\mathbb{A}}} f(v)|\theta_i|_{\mathbb{A}}$$

converges absolutely, meaning the integral

$$\int_{\mathrm{SO}_q(\mathbb{Q})\backslash \mathrm{SO}_q(\mathbb{A})} F_f(g)|dg|_{\mathbb{A}}$$

is defined.

12.4. **Using Fourier transform and Poisson summation formula.** With the same notation as in the previous subsection, in this subsection we will prove the following lemma

Lemma 138. *For $f \in \mathcal{S}(V(\mathbb{A}))$, we have*

$$\sum_{x \in \mathbb{Q}} \int_{U(i)_{\mathbb{A}}} f(v)|\theta_i|_{\mathbb{A}} = \sum_{x \in \mathbb{Q}} \int_{V(\mathbb{A})} f(v) \exp_{\mathbb{A}}(q(v)x)|dv|_{\mathbb{A}},$$

where $|dv|_{\mathbb{A}}$ is the Tamagawa measure on $V(\mathbb{A}) = \mathbb{A}^n$.

Proof. We first define a continuous map ϕ_f on $\mathbb{A}$ by

$$\phi_f(i) := \int_{U(i)_{\mathbb{A}}} f|\theta_i|_{\mathbb{A}}.$$

By using (8) from § 12.2.3, we find that the Fourier transform of ϕ_f is

$$\begin{aligned}
\widehat{\phi_f}(x) &= \int_{\mathbb{A}} \phi_f(y) \exp_{\mathbb{A}}(yx)|dy|_{\mathbb{A}}, \\
&= \int_{\mathbb{A}} \left(\int_{U(i)_{\mathbb{A}}} f \exp_{\mathbb{A}}(ix)|d\theta_i|_{\mathbb{A}} \right) |di|_{\mathbb{A}}, \\
&= \int_{V(\mathbb{A})} f(v) \exp_{\mathbb{A}}(q(v)x)|dv|_{\mathbb{A}}.
\end{aligned}$$

From this, we can deduce Poisson summation formula for ϕ_f (one needs to check certain analytic conditions of ϕ_f in order to have the Poisson summation formula, we refer to [Wei65, Proposition 1,2] or [Igu78, Chapter 4] for more details on this), giving

$$\sum_{x \in \mathbb{Q}} \phi_f(x) = \sum_{x \in \mathbb{Q}} \widehat{\phi_f}(x).$$

This proves the lemma. □

Lemma 139. *For $t \in \mathbb{A}^\times$, we define $f_t(x) := f(xt)$. Then*

$$I(f_t) = |t|_{\mathbb{A}}^{-n} I(\hat{f}_{t^{-1}}).$$

In particular, letting $t = 1$, we obtain $I(f) = I(\hat{f})$.

Proof. For $f \in \mathcal{S}(V(\mathbb{A}))$, let χ be a unitary character on $\mathbb{A}$ such that the bilinear map $\chi(q(x, y))$ defines an isomorphism between $V(\mathbb{A})$ and its Pontryagin dual and such that the discrete subgroup

$V(\mathbb{Q})$ of $V(\mathbb{A})$ is identified with the unitary characters on $V(\mathbb{Q})\backslash V(\mathbb{A})$ via $x \mapsto (y \mapsto \chi(q(x,y)))$. With this choice, we can consider the 'twisted' Fourier transform of f to be

$$\widehat{f}(y) = \int_{V(\mathbb{A})} f(x)\chi(q(x,y))|dx|_{\mathbb{A}},$$

where $|dx|_{\mathbb{A}}$ is the Tamagawa measure on $V(\mathbb{A})$. With this respect to this Fourier transform, the Tamagawa measure is self-dual, giving us the Fourier inversion formula

$$f(y) = \int_{V(\mathbb{A})} \widehat{f}(x)\overline{\chi(q(x,y))}|dx|_{\mathbb{A}}$$

and the Poisson summation formula. For any $g \in \mathrm{GL}(V(\mathbb{A}))$, we have $q(xg,y) = q(x,yg^t)$. Therefore, letting $f_g(x) := f(xg)$, we find

$$\begin{aligned}
\widehat{f_g}(x) &= \int_{V(\mathbb{A})} f(yg)\chi(q(x,y))|dy|_{\mathbb{A}}, \\
&= |\det(g)|_{\mathbb{A}}^{-1} \int_{V(\mathbb{A})} f(y)\chi(q(x,yg^{-1}))|dy|_{\mathbb{A}}, \\
&= |\det(g)|_{\mathbb{A}}^{-1} \int_{V(\mathbb{A})} f(y)\chi(q(xg^{-t},y))|dy|_{\mathbb{A}} \\
&= |\det(g)|_{\mathbb{A}}^{-1} \hat{f}(xg^{-t}).
\end{aligned}$$

By applying the Poisson summation formula for f_g, we find

$$\sum_{x \in V(\mathbb{Q})} f(xg) = |\det(g)|_{\mathbb{A}}^{-1} \sum_{x \in V(\mathbb{Q})} \widehat{f}(xg^{-t}).$$

For $t \in \mathbb{A}^{\times}$, we have

$$\begin{aligned}
I(f_t) &= \int_{\mathrm{SO}_q(\mathbb{A})\backslash \mathrm{SO}_q(\mathbb{A})} \left(\sum_{x \in V(\mathbb{Q})} f(xgt) \right) |dg|_{\mathbb{A}}, \\
&= |t|_{\mathbb{A}}^{-n} \int_{\mathrm{SO}_q(\mathbb{A})\backslash \mathrm{SO}_q(\mathbb{A})} \left(\sum_{x \in V(\mathbb{Q})} \widehat{f}(xgt^{-1}) \right) |dg|_{\mathbb{A}}, \\
&= |t|_{\mathbb{A}}^{-n} I(\hat{f}_{t^{-1}}).
\end{aligned}$$

□

12.5. **Final touch.** From Lemma 138 and § 12.3, we obtain the following identity

$$I(f) := \int_{\mathrm{SO}_q(\mathbb{Q})\backslash \mathrm{SO}_q(\mathbb{A})} \left(\sum_{x \in V(\mathbb{Q})} f(xg) \right) |dg|_{\mathbb{A}} = f(0)\tau_{\mathbb{Q}}(\mathrm{SO}_q) + 2\sum_{x \in \mathbb{Q}} \int_{V(\mathbb{A})} f(v)\exp_{\mathbb{A}}(q(v)x)|dv|_{\mathbb{A}},$$

for any $f \in \mathcal{S}(V(\mathbb{A}))$. This is referred to as the Siegel formula for the orthogonal groups ([Wei65, Igu78, Mar66]).

From Lemma 139, we know $I(f_t) = |t|_{\mathbb{A}}^{-n} I(\hat{f}_{t^{-1}})$, yielding

$$\begin{aligned} I(f_t) &= |t|_{\mathbb{A}}^{-n} I(\hat{f}_{t^{-1}}), \\ &= |t|_{\mathbb{A}}^{-n} \hat{f}(0)\tau_{\mathbb{Q}}(\mathrm{SO}_q) + 2|t|_{\mathbb{A}}^{-n} \sum_{x\in\mathbb{Q}} \int_{V(\mathbb{A})} \hat{f}(t^{-1}v) \exp_{\mathbb{A}}(q(v)x)|dv|_{\mathbb{A}}, \\ &= |t|_{\mathbb{A}}^{-n} \hat{f}(0)\tau_{\mathbb{Q}}(\mathrm{SO}_q) + 2 \sum_{x\in\mathbb{Q}^\times} \int_{V(\mathbb{A})} \hat{f}(v) \exp_{\mathbb{A}}(q(v)t^2x)|dv|_{\mathbb{A}} + 2f(0). \end{aligned}$$

One can show that the sum over $\mathbb{Q}^\times$ converges to 0 if $|t| \to \infty$. Hence, we find

$$\lim_{|t|\to\infty} I(f_t) = 2f(0).$$

On the other hand, we also have

$$I(f_t) = f(0)\tau_{\mathbb{Q}}(\mathrm{SO}_q) + 2\sum_{x\in\mathbb{Q}^\times} \int_{U(i)_{\mathbb{A}}} f_t(v)|\theta_i|_{\mathbb{A}} + |t|^{2-n} \int_{U(0)_{\mathbb{A}}} f(v)|\theta_0|_{\mathbb{A}}.$$

The above equality follows from § 12.3 and from our analysis of measure $|\theta_0|_{\mathbb{A}}$ on $U(0)_{\mathbb{A}}$.

One can show that the sum over $\mathbb{Q}^\times$ in the above equation is $O(|t|^{-N})$ for any positive integer N as $|t| \to \infty$ (see [Mar66, p. 136]). Thus, we have

$$\lim_{|t|\to\infty} I(f_t) = f(0)\tau_{\mathbb{Q}}(\mathrm{SO}_q).$$

This concludes the proof that $\tau_{\mathbb{Q}}(\mathrm{SO}_q) = 2$, as desired.

13. Tamagawa number over function fields

The goal of this section is to describe the Tamagawa number of a simply connected semisimple group G over function field of a curve over $\mathbb{F}_q$ as a weight count on $\mathrm{Bun}_G(X)$.

Throughout this section, we will denote by X to be a smooth projective curve over $\mathbb{F}_q$ and by G a smooth affine group scheme over X.

13.1. **Function fields.** In this section, we aim to describe $\mathbb{P}^1$ over a field k as a smooth, projective, geometrically connected algebraic curve. We then define the function field $k_{\mathbb{P}^1}$ of $\mathbb{P}^1$ and completions of $k_{\mathbb{P}^1}$.

The projective k-scheme $\mathbb{P}^1_k := \mathrm{Proj}\ k[x_0, x_1]$ can be described as follows:

(1) The points of $\mathbb{P}^1$ consist of homogeneous prime ideals of the $\mathbb{Z}$-graded ring $k[x_0, x_1]$ [16].
(2) For a homogeneous polynomial of positive degree $f \in k[x_1, x_2]$, let $D(f)$ be the set of homogeneous prime ideals of $k[x_0, x_1]$ not containing f. These sets form a basis of open sets for $\mathbb{P}^1$. Furthermore, one can think of $D(f)$ as $\mathrm{Spec}(k[x_0, x_1]_f)_0$, the spectrum of the algebra of elements in $k[x_0, x_1]_f$ having degree 0. For example, one can identify $D(x_i)$ with the affine scheme $\mathrm{Spec}\, k[x_{0/i}, x_{1/i}]/(x_{i/i} - 1)$, where $x_{i/j}$ is identified with x_i/x_j.
(3) The structure sheaf of $\mathbb{P}^1$ is obtained by giving $D(f)$ the structure sheaf of $\mathrm{Spec}(k[x_0, x_1]_f)_0$. In particular, $\mathcal{O}_{\mathbb{P}^1}(D(x_0)) = k[x_{1/0}], \mathcal{O}_{\mathbb{P}^1}(D(x_1)) = k[x_{0/1}]$ and the gluing of the structure sheaves on $D(x_0) \cap D(x_1) = D(x_0 x_1)$ is obtained by sending $x_{0/1} \mapsto x_{1/0}$.

Being an integral scheme (i.e. $\mathcal{O}_{\mathbb{P}^1}(U)$ is an integral domain for every nonempty subset U of $\mathbb{P}^1$), $\mathbb{P}^1$ has a generic point η (i.e. a point that is dense in $\mathbb{P}^1$), corresponding to the homogeneous prime ideal (0) in $k[x_0, x_1]$ (because every open set $D(f)$ contains (0)). The stalk $\mathcal{O}_{\mathbb{P}^1,\eta}$ of $\mathbb{P}^1$ at η, and hence the residue field $\kappa(\eta)$, is noncanonically isomorphic to $k(T)$ (i.e. if we view η as an element of $\mathrm{Spec}\, k[x_{0/1}] \hookrightarrow \mathbb{P}^1$, its stalk is then $k[x_{0/1}]_{(0)} = k(x_{0/1})$). We denote this as $k_{\mathbb{P}^1}$ and call it the *function field* of $\mathbb{P}^1$ over k.

Proposition 140. *Closed points of $\mathbb{P}^1$ are in bijection with completions of $k_{\mathbb{P}^1}$.*

Proof. A point in $\mathbb{P}^1$ is closed if it is closed in each open set $D(x_i)$ of $\mathbb{P}^1$. Furthermore, a point in an affine scheme $\mathrm{Spec}\, A$ is closed if it corresponds to a maximal ideal in A, and the maximal ideals of $k[x]$ are in bijection with monic irreducible polynomials in $k[x]$. Thus, closed points of $\mathbb{P}^1$ corresponds to homogeneous polynomials $Q(x_0, x_1) \in k[x_0, x_1]$ so that either $Q(x_{0/1}, 1)$ is monic irreducible in $x_{0/1}$ or $Q(1, x_{1/0})$ is monic irreducible in $x_{1/0}$.

For a closed point $x \in \mathbb{P}^1$ corresponding to a homogeneous polynomial $Q(x_0, x_1) \in k[x_0, x_1]$, the stalk $\mathcal{O}_{\mathbb{P}^1,x}$ of $\mathbb{P}^1$ at x is $k[x_{0/1}]_{(Q(x_{0/1},1))} \cong k[x_{1/0}]_{(Q(1,x_{1/0}))}$. The residue field $\kappa(x)$ at x is then noncanonically isomorphic to $k[t]/(Q(t, 1))$, which is a finite extension of k. We denote by $\mathcal{O}_x$ the completion of the local ring $\mathcal{O}_{\mathbb{P}^1,x}$, then $\mathcal{O}_x$ is a complete discrete valuation ring with residue field $\kappa(x)$, noncanonically isomorphic to the power series ring $\kappa(x)[[t]]$. Let k_x be the fraction field of $\mathcal{O}_x$. We find that k_x is a completion of $k_{\mathbb{P}^1}$. □

13.2. **Integral model.** Let G_0 be a linear algebraic group over k_X.

Definition 141. *An integral model of G_0 is an affine and smooth group scheme $\pi : G \to X$ whose generic fiber* [17] *is isomorphic to G_0.*

Example 142. An integral model $\overline{\mathrm{SL}_2} \to \mathbb{P}^1$ for SL_2 can be obtained by base change $\overline{\mathrm{SL}_2} = \mathrm{SL}_2 \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathbb{P}^1$.

[16] an ideal of $k[x_0, x_1]$ is homogeneous if it is generated by homogeneous polynomials

[17] i.e. let η be a generic point of X then we have a morphism $\mathrm{Spec}\,\kappa(\eta) \to \mathrm{Spec}\,\mathcal{O}_{X,x} \to X$, giving us the generic fiber $G \times_X \mathrm{Spec}\,\kappa(\eta)$ as a scheme over $\kappa(\eta)$

Given such a group scheme, for every commutative ring R equipped with a map $u : \mathrm{Spec}(R) \to X$, we can associate a group $G(R)$. If u factors through the generic point η of X, we can equip R with the structure of k_X-algebra via u, and $G(R)$ can be identified with $G_0(R)$. A choice of integral model gives additional structures:

(1) For each closed point $x \in X$, we have a morphism $\mathrm{Spec}\,\mathcal{O}_x \to \mathrm{Spec}\,\mathcal{O}_{X,x} \to X$ [18], so we can consider the group $G(\mathcal{O}_x)$ of $\mathcal{O}_x$-valued points of G.
(2) For a closed point $x \in X$, we have a morphism $\mathrm{Spec}\,\kappa(x) \to \mathrm{Spec}\,\mathcal{O}_{X,x} \to X$, so we can consider the group $G(\kappa(x))$ of $\kappa(x)$-valued point of G. We have a surjective map $G(\mathcal{O}_x) \to G(\kappa(x))$ because G is smooth [19].
(3) For a finite set S of closed points of X, we have a morphism $\mathrm{Spec}\,\mathbb{A}_X^S \to X$ so we can consider the group $G(\mathbb{A}_X^S)$ of $\mathbb{A}_X^S$-valued point of G. It is an open subgroup of $G(\mathbb{A}_X) = G_0(\mathbb{A}_X)$ and as $\mathrm{Spec}\,\mathbb{A}_X^S = \prod_{x\in S} \mathrm{Spec}\,k_x \times \prod_{x\notin S} \mathrm{Spec}\,\mathcal{O}_x$, $G(\mathbb{A}_X^S)$ is isomorphic to the direct product $\prod_{x\in S} G(k_x) \times \prod_{x\notin S} G(\mathcal{O}_x)$.

13.3. **Adelic uniformation of G-bundles.** In this section, we will describe the set of isomorphism classes of G-bundles in terms of a double quotient space.

Definition 143. *Let Y be a X-scheme. A G-*bundle *on Y is a Y-scheme P equipped with an action of $G_Y := G \times_X Y$ given by $G_Y \times_Y P \cong G \times_X P \to P$ which is locally trivial in the following sense: there exists an open immersion $U \to Y$ and a G_Y-equivariant isomorphism $U \times_Y P \cong U \times_Y G_Y$.*

Let $\mathrm{Bun}_G(Y)$ be the groupoid of G-bundles on Y whose morphism are isomorphims of G-bundles.

For a G-bundle P on X and an open covering $U \to X$, we denote by $P|_U$ the pullback of P along $U \to X$.

Theorem 144 (Adelic uniformisation theorem)**.** *Let X be an algebraic curve over $\mathbb{F}_q$ and G be a smooth affine group scheme over X. Assume that the fibers of G are connected and that the generic fiber of G is semisimple and simply connected. Then*

(a) There is a bijection between the double quotient

$$G(k_X)\backslash G(\mathbb{A}_X)/G(\mathbb{A}_X^{\emptyset})$$

and the set of isomorphism classes of G-bundles on X, sending $\gamma \in G(\mathbb{A}_X)$ to a G-bundle P_γ.

(b) For $\gamma \in G(\mathbb{A}_X)$ then the automorphism group of P_γ corresponds to elements in $\gamma^{-1}G(k_X)\gamma \cap G(\mathbb{A}_X^{\emptyset})$.

Following [GL19, §1.3.2], we attempt to give a sketch for the proof of this theorem. We first remark that we have defined G-bundles in terms of the Zariski topology. However, as far as we are aware, one needs G-bundles in flat/etale topology for this result to hold. Due to our ignorance in this topic, we will pretend that we know what 'flat/etale topology' means and accept any results about G-bundles in 'flat/etale topology' which are used in the below proof of the theorem.

Sketch. For convenience, for $x \in X$, we denote $D_x = \mathrm{Spec}\,\mathcal{O}_x$ and $D_x^* = \mathrm{Spec}\,k_x$.

An element $\gamma \in G(\mathbb{A}_X)$ can be identified with $(\gamma_x \in G(k_x))_{x\in X}$ where $\gamma_x \in G(\mathcal{O}_x)$ for all but a set S of finitely many closed points of X. Hence, by Beauville-Laszlo theorem on gluing G-bundles, there exists a G-bundle P_γ on X such that

[18]For example, let $X = \mathrm{Spec}\,\mathbb{Z}$, a prime number in $\mathbb{Z}$ corresponds to a closed point of X, we find $\mathcal{O}_{X,p} = \mathbb{Z}_{(p)}$ is a local ring with maximal ideal $\mathfrak{m}_{X,p} = p\mathbb{Z}_{(p)}$. The completion of $\mathcal{O}_{X,p}$ with respect to this maximal ideal is $\mathcal{O}_p = \mathbb{Z}_p$. Thus, we have $\mathbb{Z} \to \mathbb{Z}_{(p)} \to \mathbb{Z}_p$, giving us $\mathrm{Spec}\,\mathcal{O}_x \to \mathrm{Spec}\,\mathcal{O}_{X,x} \to X$

[19]Smoothness implies a condition on the Jacobian of the local coordinates at a point, and by a generalisation of Hensel's lifting lemma, we have surjectivity. For more details, see p.20 of See Weil's book Adeles and Algebraic Groups

(1) P_γ is trivialised over $U = X - S$ via some $\phi : G \times_X U \xrightarrow{\sim} P_\gamma|_U$.
(2) For each $x \in S$, P_γ is trivialised over D_x via $\psi_x : G \times_X D_x \xrightarrow{\sim} P_\gamma|_{D_x}$.
(3) As $D_x^* \hookrightarrow D_x$ and $D_x^* \hookrightarrow U$, the two trivialisations are glued together on $\operatorname{Spec} k_x$ by $\gamma_x \in G(k_x)$ for $x \in S$. In other words, the map

$$G(k_x) \xrightarrow[\sim]{\phi|_{D_x^*}} \Gamma(D_x^*, P|_{D_x^*}) \xrightarrow[\sim]{\psi_x^{-1}|_{D_x^*}} G(k_x) \tag{18}$$

is given by sending $1 \mapsto \gamma_x$.

Note that the definition of P_γ is independent of the choice of S, as long as S contains all points x so $\gamma_x \notin G(\mathcal{O}_x)$.

Conversely, considering a G-bundle $\mathcal{E}$, we will show that there exists $\gamma \in G(\mathbb{A}_X)$ such that $\mathcal{E} = P_\gamma$. Because the fibers of G are connected, by a theorem of Lang, for each closed point $x \in X$, the bundle $\mathcal{E}$ is trivial over $\operatorname{Spec}(\kappa(x))$. Because G is smooth over X, $\mathcal{E}$ is also smooth over X. By Hensel's lemma, any trivialisation of P over $\operatorname{Spec}(\kappa(x))$ can be extended to a trivialisation of P over $\operatorname{Spec}(\mathcal{O}_x)$. This implies $\mathcal{E}$ is trivial over $\operatorname{Spec}(k_x)$ for any $x \in X$. Therefore, by a theorem of Harder, as the generic fiber of G is connected, semisimple and simply connected, $\mathcal{E}$ is trivial over the generic point. By a direct limit argument, $\mathcal{E}$ is trivial over some open subset $U \subset X$.

Let S be the set of closed points of X that are not contained in U. We know that $\mathcal{E}$ is trivial over U and over D_x for all $x \in S$. By the previous construction, the gluing data gives us an element $\gamma \in G(\mathbb{A}_X)$, so $\mathcal{E} = P_\gamma$.

Thus, we have constructed a surjective map $G(\mathbb{A}_X) \to |\operatorname{Bun}_G(X)|$ sending $\gamma \mapsto P_\gamma$.

Finally, we identify when P_γ is isomorphic to $P_{\gamma'}$ for $\gamma, \gamma' \in G(\mathbb{A}_X)$. Both are trivialised at the generic point, implying the isomorphism $P_\gamma|_{\operatorname{Spec} k_X} \xrightarrow{\sim} P_{\gamma'}|_{\operatorname{Spec} k_X}$ corresponds to an element in $\operatorname{Aut}(G \times_X \operatorname{Spec} k_X)$, corresponding to an element $\alpha \in G(k_X)$. In particular, we have

$$\Gamma(\operatorname{Spec} k_X, P_\gamma) \xrightarrow[\sim]{\phi_\gamma^{-1}} G(k_X) \xrightarrow{\times\alpha} G(k_X) \xrightarrow[\sim]{\phi_{\gamma'}} \Gamma(\operatorname{Spec} k_X, P_{\gamma'}),$$

where $\phi_\gamma : G \times_X \operatorname{Spec} k_X \xrightarrow{\sim} P_\gamma|_{\operatorname{Spec} k_X}$ and $\phi_{\gamma'} : G \times_X \operatorname{Spec} k_X \xrightarrow{\sim} P_{\gamma'}|_{\operatorname{Spec} k_X}$ are trivialisations of P_γ and $P_{\gamma'}$ over $\operatorname{Spec} k_X$, respectively.

A trivialisation at the generic point induces a trivialisation over some open set U of X. Let S be a set of closed points not contained in U. We know P_γ and $P_{\gamma'}$ are also trivialised over D_x for $x \in S$. Arguing similarly, $P_\gamma|_{D_x} \xrightarrow{\sim} P_{\gamma'}|_{D_x}$ corresponds to an element $\beta_x \in G(\mathcal{O}_x)$ for all $x \in S$. In particular, for $x \in S$, we have

$$\Gamma(D_x, P_\gamma) \xrightarrow[\sim]{\psi_{\gamma,x}^{-1}} G(\mathcal{O}_x) \xrightarrow{\times\beta_x} G(\mathcal{O}_x) \xrightarrow[\sim]{\psi_{\gamma',x}} \Gamma(D_x, P_{\gamma'})$$

Lastly, we know the gluing data of trivialisations of P_γ over U and over D_x for $x \in S$ comes from $\gamma_x \in G(k_x)$, i.e. (18). Similarly for γ'. Thus, by combining everything, we find

$$\gamma^{-1}\alpha\gamma' = \prod_{v \in S} \beta_x^{-1} \in G(\mathbb{A}_X^{\emptyset}).$$

Thus, an element in $\gamma^{-1}G(k_X)\gamma' \cap G(\mathbb{A}_X^{\emptyset})$ defines an isomorphism between P_γ and $P_{\gamma'}$ uniquely. We are done. □

13.4. **Tamagawa number in terms of G-bundles.** Let X be an algebraic curve over $\mathbb{F}_q$ and k_X be the function field of X. Let G_0 be a connected semisimple simply connected linear algebraic group. In this section, we will give a formula for the Tamagawa number of G_0 in terms of G-bundles.

Let $\pi : G \to X$ be an integral model of G_0 (such an integral model always exists), let $\Omega_{G/X}$ be the relative cotangent bundle of π. Then $\Omega^n_{G/X} := \bigwedge^n \Omega_{G/X}$ is a line bundle on G, where $n = \dim(G_0)$. Let $\mathcal{L}$ be the pullback of $\Omega^n_{G/X}$ along the identity section $e : X \to G$. Sections of $\mathcal{L}$ can be identified with left-invariant differential forms on G via the canonical isomorphism $\pi^*\mathcal{L} \cong \Omega^n_{G/X}$ (see [BLR90, p. 100]). Let $\mathcal{L}_0 := \operatorname{Spec} k_X \times_X \mathcal{L}$ be the generic fiber of $\mathcal{L}$, whose global sections form a 1-dimensional k_X-vector space. A non-zero global section ω of $\mathcal{L}_0$ can be viewed as a global left-invariant nowhere-vanishing algebraic differential form on G_0. For every closed point $x \in X$, ω induces a left-invariant Haar measure $d\mu_{\omega,x}$ on $G(k_x)$.

For an invertible sheaf $\mathcal{L}$ of $\mathcal{O}_X$-modules and a nonzero global section ω of its generic fiber $\mathcal{L}_0$, one can associate a divisor on X as follows. For a closed point $x \in X$, we consider the stalk $\mathcal{L}_x \subset \mathcal{L}_0$ at x, which is a $\mathcal{O}_x$-module of rank 1 inside a 1-dimensional k_X-vector space. Then $\omega\mathcal{O}_x$ is also a rank 1 $\mathcal{O}_x$-submodule of $\mathcal{L}_0$. Let $t_x \in \mathcal{O}_x$ be a uniformiser element then $\omega\mathcal{O}_x = t_x^{-n_x}\mathcal{L}_x$ for some integer n_x. We define $v_x(\omega) := n_x$ to be the *order of vanishing* of ω at x.

Lemma 145. *For every closed point $x \in X$, We have*

$$\mu_{\omega,x}(G(\mathcal{O}_x)) = \frac{|G(\kappa(x))|}{|\kappa(x)|^{n+v_x(\omega)}},$$

where $v_x(\omega) \in \mathbb{Z}$ denotes the order of vanishing of ω at x.

Sketch. If we view ω as a left-invariant differential form on $G(k_x)$ via the isomorphism $\pi^*\mathcal{L} \cong \Omega^n_{G/X}$, $v_x(\omega)$ can be described as follows. At the neighborhood U of the identity e of $G(k_x)$, ω can be written as $\omega = f(t)dt_1 \wedge \cdots \wedge dt_n$ where $t_1, \ldots, t_n$ are the local coordinates at e, and $f : U \to k_x$ is an invertible rational function. Then $f(e)\mathcal{O}_x = t_x^{-v_x(\omega)}\mathcal{O}_x$. In other words, the image of $\omega(e) \in \bigwedge^n T_e^*(G(\mathcal{O}_x))$ under $\bigwedge^n T_e(G(\mathcal{O}_x))$ generates a fractional ideal $\mathfrak{p}^{-v_x(\omega)}$ of k_x. Because ω is left-invariant so for any $g \in G(\mathcal{O}_x)$, the image of $\omega(g) \in \bigwedge^n T_g^*(G(\mathcal{O}_x))$ under $\bigwedge^n T_g(G(\mathcal{O}_x))$ generates the fractional ideal $\mathfrak{p}^{-v_x(\omega)}$ of k_x. In other words, under new local coordinates $y_1, \ldots, y_n$ at the neighborhood of $g \in G(\mathcal{O}_x)$, we have $\omega = f'(y)dy_1 \wedge \cdots \wedge dy_n$ where f' is a rational function so that $f'(g)\mathcal{O}_x = t_x^{-v_x(\omega)}\mathcal{O}_x$. By the definition of Weil measure, we then find that $t_x^{v_x(\omega)}\omega$ defines the Weil measure on $G(\mathcal{O}_x)$, or ω defines a measure $|\kappa(x)|^{-v_x(\omega)}\mu_{\mathrm{Weil}}$ on $G(\mathcal{O}_x)$. Thus, by Theorem 87, we find

$$\mu_{\omega,x}(G(\mathcal{O}_x)) = |\kappa(x)|^{-v_x(\omega)}\mu_{\mathrm{Weil}}(G(\mathcal{O}_x)) = \frac{|G(\kappa(x))|}{|\kappa(x)|^{n+v_x(\omega)}}.$$

□

From § 5.5.3, the Tamagawa measure $\mu_{G_0,X}$ of $G(\mathbb{A}_X) = G_0(\mathbb{A}_X)$ is

$$q^{(1-g)n} \prod_{x\in X}{}' \mu_{\omega,x}.$$

where g is the genus of X. We also have

$$\prod_{x\in X} |\kappa(x)|^{v_x(\omega)} = \prod_{x\in X} q^{\deg(x)v_x(\omega)} = q^{\sum_{x\in X}\deg(x)v_x(\omega)} = q^{\deg\mathcal{L}} = q^{\deg\Omega_{G/X}}.$$

Here $\deg(x) := [\kappa(x) : \mathbb{F}_q]$, $\deg\mathcal{L} := \sum_{x\in X}\deg(x)v_x(\omega)$ where the sum is over all closed points of X [20], $\deg\mathcal{L} = \deg\Omega_{G/X} = n$ because of the isomorphism $\pi^*\mathcal{L} \cong \Omega^n_{G/X}$. Thus, combining with the previous lemma, we find

$$\mu_{G_0,X}(G(\mathbb{A}_X^{\emptyset})) = q^{n(1-g)-\deg(\Omega_{G/X})} \prod_{x\in X} \frac{|G(\kappa(x)|}{|\kappa(x)|^n},$$

given that the above infinite product converges.

[20]this does not depend on the choice of ω because for any two nonzero global sections ω and ω' of $\mathcal{L}$, there exists a nowhere-vanishing function $f \in \mathcal{O}_X(X)$ so $\omega = f\omega'$, and one can show $\sum_{x\in X} v_x(f)\deg(x) = 0$.

Proposition 146. *Let X be an algebraic curve of genus g over a finite field $\mathbb{F}_q$ and let G be a smooth affine group scheme of dimension n over X. Suppose that the fibers of G are connected and that the generic fiber of G is semisimple and simply connected. Then the Tamagawa number of G equals*

$$\tau_{k_X}(G) = q^{n(g-1)+\deg(\Omega_{G/X})} \prod_{x \in X} \frac{|G(\kappa(x))|}{|\kappa(x)|^d} \sum_{P \in \mathrm{Bun}_G(X)} \frac{1}{|\operatorname{Aut}(P)|},$$

given that the right-hand side converges.

Proof. For each $z \in G(k_X)\backslash G(\mathbb{A}_X)/G(A_X^{\emptyset})$, denote by O_z the inverse image of z under the projection from $G(k_X)\backslash G(\mathbb{A}_X)$. Thus, we have

$$\tau_{k_X}(G) = \sum_{z \in G(k_X)\backslash G(\mathbb{A}_X)/G(A_X^{\emptyset})} \mu_{G,k_X}(O_z).$$

Let $\gamma \in G(\mathbb{A}_X)$ lie in the preimage of z, then the preimage of O_z under the projection from $G(\mathbb{A}_X)$ is

$$\bigsqcup_{\alpha \in G(k_X) \cap \gamma G(\mathbb{A}_X^{\emptyset})\gamma^{-1}} \alpha\gamma G(\mathbb{A}_X^{\emptyset}).$$

Note that $G(k_X) \cap \gamma G(\mathbb{A}_X^{\emptyset})\gamma^{-1}$ is a finite group because it is the intersection of a discrete and a compact group. It follows that

$$\mu_{G,k_X}(O_z) = \frac{\mu_{G,k_X}(\mathbb{A}_X^{\emptyset})}{|G(k_X) \cap \gamma G(\mathbb{A}_X^{\emptyset})\gamma^{-1}|}.$$

Thus, combining with the previous theorem, we find

$$\tau_{k_X}(G) = \tau_{G,k_X}(G(\mathbb{A}_X^{\emptyset})) \sum_{P \in \mathrm{Bun}_G(X)} \frac{1}{|\operatorname{Aut}(P)|}.$$

We are done. □

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