§1.4 Limits and Colimits : Page 39

- Limit:

$$\begin{tikzcd} & \varprojlim_{\mathcal{I}} A_i \arrow[dl, "f_j"'] \arrow[dr, "f_k"] & \\ A_j \arrow[rr, "F(m)"] & & A_k \end{tikzcd}$$

$\mathcal{I}$: index category (usually partially ordered set)
$m : j \to k$ morphism in $\mathcal{I}$

. Example: In cat of Sets, $\varprojlim A_i = \left\{ (a_i)_{i \in \mathcal{I}} \in \prod A_i : F(m) a_j = a_k \text{ for all } m \in \mathrm{Mor}(j,k) \right\}$ along with obvious projection maps to each A_i

$\Rightarrow$ Should view elements of limits as sequence (a_i)

- Colimit: Reverse arrows $\varinjlim A_i$

. Example: $\mathcal{I}$ filtered index cat. In sets. $\varinjlim A_i = \left\{ (a_i, i) \in \coprod_{i \in \mathcal{I}} A_i \right\} / (a_i, i) \sim (a_j, j) \Leftrightarrow$ exists $f : A_i \to A_k$ and $g : A_j \to A_k$ so $f(a_i) = g(a_j)$ in A_k

$\hookrightarrow$ check this out in Vakil note.

Main point: One can tell which object is (co)limit by looking at the set structure as described.

2.2.10 Constant presheaves and constant sheaves

X top space and S a set. Define $S_{pre}(U) = S$ for all open $U \subset X$. Then S_{pre} forms presheaf with restriction map the identity. → Constant presheaf associated to S.

This is not generally a sheaf

2.2.E Constant sheaf $\mathcal{F}(U)$ be maps to S that are locally constant, i.e. for any point p in U there is open neighborhood of p where the function is constant. → Constant sheaf associated to S

Denote as $\underline{S}$.

(endows S with the discrete topology, and let $\mathcal{F}(U)$ continuous map $U \to S$)

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2.2.11. Escape étale of a (pre)sheaf.

$\mathcal{F}$ presheaf on top space X; $\pi: F \to X$ continuous

F: disjoint union of all stalks of $\mathcal{F}$ stalk at p = set of germs at p

- $\pi: F \to X$ maps {germ at p} to p.
- Topology of F: Each $s \in \mathcal{F}(U)$ determines subset $\{(x, s_x) : x \in U\}$ of F.

These subsets form bases of topology.

$s_x \in \mathcal{F}(V)$ that $x \in V \subset U$ and $\mathrm{res}_{U,V} s = s_x$

- For each $y \in F$, there is open neighborhood V of y and open neighborhood U of $\pi(y)$ so $\pi|_V$ homeomorphism from V to U.

First, y is germ of $\mathcal{F}$ at $\pi(y)$ $\Rightarrow$ $y = (f, \text{open } U)$ so $\pi(y) \in U$, $f \in \mathcal{F}(U)$

Since $f \in \mathcal{F}(U)$, we obtain open set $\underbrace{\{(x, f_x) : x \in U\}}_{U'}$ $f_x \in \mathcal{F}(V)$ where $x \in V \subset U$ and $\mathrm{res}_{U,V} f = f_x$.

Then $\pi|_{U'}$ is homeomorphism from U' to V. It is obviously bijective.

2.2.11 Push forward sheaf or direct image sheaf.

Suppose $\pi: X \to Y$ continuous map, $\mathcal{F}$ presheaf on X. Define $\pi_* \mathcal{F}$ by $\pi_* \mathcal{F}(V) = \mathcal{F}(\pi^{-1}(V))$ where V open subset of Y. Show $\pi_* \mathcal{F}$ presheaf on Y, and is a sheaf if $\mathcal{F}$ is.

- Presheaf is clear: Since if $U \subset V$ open in Y then $\pi^{-1}(U) \subset \pi^{-1}(V)$ so exists $res_{V,U}: \pi_* \mathcal{F}(V) \to \pi_* \mathcal{F}(U)$

$$\mathcal{F}(\pi^{-1}(V)) \to \mathcal{F}(\pi^{-1}(U))$$

- Identity axiom: $\{U_i\}$ open cover of $U \subset Y$, $f_1, f_2 \in \pi_* \mathcal{F}(U) = \mathcal{F}(\pi^{-1}(U))$.

$res_{U,U_i} f_1 = res_{U,U_i} f_2 \Rightarrow res_{U,U_i} f = res_{\pi^{-1}(U), \pi^{-1}(U_i)} f$

 - $\{\pi^{-1}(U_i)\}$ open cover of $\pi^{-1}(U) \subset X$

- Gluability axiom: $\{U_i\}$ open cover of $U \subset Y \Rightarrow \{\pi^{-1}(U_i)\}$ open cover of $\pi^{-1}(U) \subset X$

.....

2.2.12

2.2.13. Ringed spaces, and $\mathcal{O}_X$-modules

$\mathcal{O}_X$ structure sheaf $\mathcal{O}_{X,p}$ stalk at p

- $\mathcal{O}_X$ sheaf of rings on topological X. Then $(X, \mathcal{O}_X)$ called ringed space. $\mathcal{O}_U$ for $U \subset X$.
- $\mathcal{O}_X$-modules over sheaf of rings $\mathcal{O}_X$: sheaf $\mathcal{F}$ of abelian groups so $\mathcal{F}(U)$ is $\mathcal{O}_X(U)$-module s.t. the action behave well wrt restriction maps.
- Sheaf of abelian group is $\underline{\mathbb{Z}}$-module, $\underline{\mathbb{Z}}$ constant sheaf associated to $\mathbb{Z}$.

$p \in X$.

2.2.J. If $(X, \mathcal{O}_X)$ ringed space, and $\mathcal{F}$ is $\mathcal{O}_X$-module, describe how $\mathcal{F}_p$ is $\mathcal{O}_{X,p}$-module

- What is ring structure of $\mathcal{O}_{X,p}$? Elements $\{(f, \text{open } U) : f \in \mathcal{O}_X(U), p \in U\}$ modulo ...

  $(f_1, \text{open } U_1) + (f_2, \text{open } U_2) = (f_1|_V + f_2|_V, \text{open } V = U_1 \cap U_2)$

  similarly for multiplication + check that it's well-defined.

- $\mathcal{F}(U)$ is $\mathcal{O}_X(U)$-module

  $\mathcal{F}_p$: elements $\{(f, \text{open } U) : f \in \mathcal{F}(U), p \in U \subset X\}$ modulo ... → eqv relation

- How $\mathcal{O}_{X,p}$ acts on $\mathcal{F}_p$? $f \in \mathcal{O}_X(U)$ $p \in U$ acts on $g \in \mathcal{F}(V)$, $p \in V$ by letting

$W = U \cap V$: action of $f|_W \in \mathcal{O}_X(W)$ on $g|_W \in \mathcal{F}(W)$, $p \in W$ → since $\mathcal{F}(W)$ is $\mathcal{O}_X(W)$-module

2.14. Motivation of $\mathcal{O}_X$-module as sheaf of sections of vector bundles.

- What is a vector bundle? May be later.

23A Morphism of (pre)sheaves induce morphisms of stalks.

$\phi: \mathcal{F} \to \mathcal{G}$ morphism of presheaves on X, $p \in X$. Describe $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$ morphism of stalks

In other words, taking stalk at p induces functor $Sets_X \to Set$.

Recall $\mathcal{F}_p = \{(f, \text{open } U) : f \in \mathcal{F}(U),\ p \in U\}$ up to equivalence relation.

Define $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$ as $(f, \text{open } U) \longmapsto (\phi f, \text{open } U)$

Check: Well-define: if $(g, V) \sim (f, U)$ on $\mathcal{F}_p$ then exists $W \subset U, V$ so $res^{\mathcal{F}}_{V,W}\, g = res^{\mathcal{F}}_{U,W}\, f$.

Need to show $(\phi g, V) \sim (\phi f, U)$ on $\mathcal{G}_p$.

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r, "\phi"] \arrow[d, "res^{\mathcal{F}}_{V,W}"'] & \mathcal{G}(V) \arrow[d, "res^{\mathcal{G}}_{V,W}"] \\
\mathcal{F}(W) \arrow[r, "\phi"'] & \mathcal{G}(W) \\
\mathcal{F}(U) \arrow[u, "res^{\mathcal{F}}_{U,W}"] \arrow[r, "\phi"'] & \mathcal{G}(U) \arrow[u, "res^{\mathcal{G}}_{U,W}"']
\end{tikzcd}$$

$$res^{\mathcal{G}}_{V,W}\, \phi g = \phi\left(res^{\mathcal{F}}_{V,W}\, g\right)$$

$$res^{\mathcal{G}}_{U,W}\, \phi f = \phi\left(res^{\mathcal{F}}_{U,W}\, f\right)$$

$=$

$\boxed{2.3B}$ $Sets_X$ category of sheaves of sets.

$\pi: X \to Y$ continuous map of top space. Show pushforward gives a functor: $\pi_*: Sets_X \to Sets_Y$

- Push forward: Given sheaf $\mathcal{F}$ on X, i.e. $\mathcal{F} \in Sets_X$, construct sheaf on Y by $\pi_*\mathcal{F}(U) := \mathcal{F}(\pi^{-1}(U))$ for open $U \subset Y$

- Recall $Sets_X$: objects are sheaves on X; morphism $f: \mathcal{F} \to \mathcal{G}$ sends $\mathcal{F}(U) \mapsto \mathcal{G}(U)$ so that

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "f"] \arrow[d, "res_{U,V} \text{ in } \mathcal{F}"'] & \mathcal{G}(U) \arrow[d, "res_{U,V} \text{ in } \mathcal{G}"] \\ \mathcal{F}(V) \arrow[r, "f"] & \mathcal{G}(V) \end{tikzcd}$$

where $V \subset U \subset X$ open.

- Describe functor $\pi_*: Sets_X \to Sets_Y$. π_* sends $\mathcal{F} \in Sets_X$ to $\pi_*\mathcal{F} \in Sets_Y$

Morphism $f: \mathcal{F} \to \mathcal{G}$ in $Sets_X$ then $\pi_* f: \pi_*\mathcal{F} \to \pi_*\mathcal{G}$ defined as

- $\pi_*\mathcal{F}(U) \mapsto \pi_*\mathcal{G}(U)$ for $U \subset Y$

Check: Composition, Identity

2.3c) $\mathcal{F}$ presheaf, $\mathcal{G}$ sheaf of sets on X. $\mathcal{H}om(\mathcal{F},\mathcal{G})(U) := \mathrm{Mor}(\mathcal{F}|_U, \mathcal{G}|_U)$

- What is presheaf structure on $\mathcal{H}om(\mathcal{F},\mathcal{G})$?
  - What is restriction map from $\mathcal{H}om(\mathcal{F},\mathcal{G})(U) \to \mathcal{H}om(\mathcal{F},\mathcal{G})(V)$ for $V \subset U \subset X$?

Denote $res^{Hom}_{U,V}$ as this map

Recall: $\mathrm{Mor}(\mathcal{F}|_U, \mathcal{G}|_U)$ as morphism of presheaves $f: \mathcal{F}|_U \to \mathcal{G}|_U$ consists of maps $W \subset U$
$f(W): \mathcal{F}|_U(W) = \mathcal{F}(W) \to \mathcal{G}|_U(W) = \mathcal{G}(W)$ such that restriction maps in each $\mathcal{F}|_U, \mathcal{G}|_U$ behave ...

$$\begin{tikzcd} \mathcal{F}(W) \arrow[r, "f(W)"] \arrow[d, "res^{\mathcal{F}}_{W,Z}"'] & \mathcal{G}(W) \arrow[d, "res^{\mathcal{G}}_{W,Z}"] \\ \mathcal{F}(Z) \arrow[r, "f(Z)"'] & \mathcal{G}(Z) \end{tikzcd}$$

If $f \in \mathcal{H}om(\mathcal{F},\mathcal{G})(U)$, what is $res^{Hom}_{U,V} f \in \mathcal{H}om(\mathcal{F},\mathcal{G})(V)$?

$(res^{Hom}_{U,V} f)(W) := f(W)$ for $W \subset V \subset U \subset X$.

- Check: Commutativity of res from inclusions of sets follows from

$$\begin{tikzcd} \mathcal{H}om(\mathcal{F},\mathcal{G})(U) \arrow[r, "res_{U,V}"] \arrow[dr, "res_{U,W}"'] & \mathcal{H}om(\mathcal{F},\mathcal{G})(V) \arrow[d, "res_{V,W}"] \\ & \mathcal{H}om(\mathcal{F},\mathcal{G})(W) \end{tikzcd}$$

follows from how we define $res^{Hom}_{U,V}$.

⊛ Check identity axiom: $\{U_i\}$ open cover of $U \subset X$ and $f_1, f_2 \in \mathcal{H}om(\mathcal{F},\mathcal{G})(U)$ so $res_{U,U_i} f_1 = res_{U,U_i} f_2$ for all i. Show $f_1 = f_2$. $\Leftrightarrow f_1(W) = f_2(W)$ for any $W \subset U$. → note: one can always divide $\{U_i\}$ into smaller parts s.t. some is open cover of W while satisfying the hypothesis

$$res_{U,U_i} f_1 = res_{U,U_i} f_2 \in \mathcal{H}om(\mathcal{F},\mathcal{G})(U_i) \Rightarrow f_1(U_i) = f_2(U_i)$$

Recall: $\mathcal{G}$ is a sheaf, mean if $g_1, g_2 \in \mathcal{G}(U)$ and $res^{\mathcal{G}}_{U,U_i} g_1 = res^{\mathcal{G}}_{U,U_i} g_2$ then $g_1 = g_2$.

$$\begin{tikzcd}
\mathcal{F}(W) \arrow[r, "f(W)"] \arrow[d, "res^{\mathcal{F}}_{W,U_i}"'] \arrow[dr, phantom, "(*)"] & \mathcal{G}(W) \arrow[d, "res^{\mathcal{G}}_{W,U_i}"] \\
\mathcal{F}(U_i) \arrow[r, "f(U_i)"'] & \mathcal{G}(U_i)
\end{tikzcd}$$

$*$ $f_1(W) = f_2(W) \Leftrightarrow f_1(W)w = f_2(W)w \in \mathcal{G}(W)$ for any $w \in \mathcal{F}(W)$

We know $res^{\mathcal{G}}_{U,U_i} f_1(W)w \overset{(*)}{=} f_1(U_i)\, res^{\mathcal{F}}_{U,U_i} w$

$= f_2(U_i)$ ______

$\overset{(*)}{=} res^{\mathcal{G}}_{U,U_i} f_2(W)w$

Since $\mathcal{G}$ is a sheaf so $f_1(W)w = f_2(W)w$ for any $w \in \mathcal{F}(W)$ ⌟

Thus, $f_1(W) = f_2(W)$ for any $W \subset U \Rightarrow f_1 = f_2$ □

Check gluability axiom: $\{U_i\}$ open cover of U; $f_i \in \mathcal{H}om(\mathcal{F},\mathcal{G})(U_i)$ so that $res^{\mathcal{H}om}_{U_i,U_i\cap U_j} f_i = res^{\mathcal{H}om}_{U_j,U_j\cap U_i} f_j$. $\Rightarrow f_i(U_i\cap U_j) = f_j(U_i\cap U_j)$. Need $f\in \mathcal{H}om(\mathcal{F},\mathcal{G})(U)$ so $res^{\mathcal{H}om}_{U,U_i} f = f_i$.

- $res^{\mathcal{H}om}_{U,U_i} f = f_i \Leftrightarrow f(W) = f_i(W)$ for any $W \subset U_i$. How to define $f(W)w$ for any $W\subset U$, $w \in \mathcal{F}(W)$

  - First, suppose we have an open cover of W in $\{U_i\}$. If not, we can create more sheaves $f_i|_W \in \mathcal{H}om(\mathcal{F},\mathcal{G})(W)$ by choosing $W \subset U_i$ while satisfying the conditions.

$$\begin{tikzcd} \mathcal{F}(U_i) \arrow[r, "f(U_i)"] \arrow[d, "res_{U_i,W}"'] & \mathcal{G}(U_i) \arrow[d, "res_{U_i,W}"] \\ \mathcal{F}(W) \arrow[r, "f(W)"'] & \mathcal{G}(W) \end{tikzcd}$$

  - If there exists such f then $res^{\mathcal{G}}_{W,U_i} f(W)w = f(U_i)\, res^{\mathcal{F}}_{W,U_i} w$
$= f_i(U_i)\, res^{\mathcal{F}}_{W,U_i} w$

$\Rightarrow$ This suggests us to define $f(W)w$ as follows:

Consider $f_i(U_i)\, res^{\mathcal{F}}_{W,U_i} w \in \mathcal{G}(U_i)$, note that

$$res^{\mathcal{G}}_{U_i,U_j\cap U_i} f_i(U_i)\, res^{\mathcal{F}}_{W,U_i} w = f_i(U_i\cap U_j)\, res^{\mathcal{F}}_{W,U_i\cap U_j} w = f_j(U_i\cap U_j)\, res^{\mathcal{F}}_{U_j,U_i\cap U_j} res^{\mathcal{F}}_{W,U_j} w$$
$$= res^{\mathcal{G}}_{W,U_j} f_j(U_j)\, res^{\mathcal{F}}_{W,U_j} w$$

$\Rightarrow$ By gluability of $\mathcal{F}$, exists a unique $f(W)w \in \mathcal{G}(W)$ so $res^{\mathcal{G}}_{W,U_i} f(W)w = f_i(U_i)\, res^{\mathcal{F}}_{W,U_i} w$.

(*) Thus, we have define $f(W)$ for any $W \subset U$. Next we need to check commutative diagram

$$\begin{tikzcd} \mathcal{F}(W) \arrow[r, "f(W)"] \arrow[d, "res^{\mathcal{F}}_{W,V}"'] & \mathcal{G}(W) \arrow[d, "res^{\mathcal{G}}_{W,V}"] \\ \mathcal{F}(V) \arrow[r, "f(V)"'] & \mathcal{G}(V) \end{tikzcd}$$

$V \subset W \subset U$, If $w \in \mathcal{F}(W)$, we need $res^{\mathcal{G}}_{W,V} f(W)w = f(V) res^{\mathcal{F}}_{W,V} w . \in \mathcal{G}(V)$.

- One can suppose $V = U_i$ by applying res_{V,U_i} to both side (also thanks to identity axiom of sheaf $\mathcal{G}$).
- The above is then just: $f_i(U_i) res^{\mathcal{F}}_{W,U_i} w = f(U_i) res^{\mathcal{F}}_{W,U_i} w$, which is true as long as we can show $f_i(U_i) = f(U_i)$ or more generally:

(*) $res^{Hom}_{U,U_i} f = f_i \Leftrightarrow f(W) = f_i(W)$ for any $W \subset U_i . \Leftrightarrow f(W)w = f_i(W)w$ for any $W \subset U_i$, $w \in \mathcal{F}(W)$

This holds according to definition of $f(W)w$, i.e. one can add extra open set W in $\{U_i\}$ with extra $f_W := f_i|_W$ (while still satisfying all the conditions and $f(W)w$ is determined uniquely even if you do this. (this thanks to the identity axiom of $\mathcal{G}$). $f(W)w$ will then satisfies $f(W)w = res^{\mathcal{G}}_{W,W} f(W)w$

$$= f_i(W) res^{\mathcal{F}}_{W,W} w$$

$$= f_i(W) w$$

⊛ Hom contravariant functor in its first argument and covariant functor in its second argument.

$\mathcal{H}om(-,\mathcal{F}) : Sets_X \to Sets_X$ sends $\mathcal{G} \mapsto \mathcal{H}om(\mathcal{G},\mathcal{F})$ and morphism $f:\mathcal{G}\to\mathcal{H}$ to
$F: \mathcal{H}om(\mathcal{H},\mathcal{F}) \to \mathcal{H}om(\mathcal{G},\mathcal{F})$ defined as: $f:\mathcal{G}\to\mathcal{H}$ consists of maps $f(U): \mathcal{G}(U)\to\mathcal{H}(U)$ which will induce morphism of sheaves $f|_U : \mathcal{G}|_U \to \mathcal{H}|_U$.

Then F will consists of maps $F(U) : Mor(\mathcal{H}|_U, \mathcal{F}|_U) \to Mor(\mathcal{G}|_U, \mathcal{F}|_U)$

$$g \longmapsto g\circ f|_U$$

$$(g(W):\mathcal{H}(W)\to\mathcal{F}(W)) \mapsto (g\circ f|_U)(W): \mathcal{G}(W) \xrightarrow{f(W)} \mathcal{H}(W) \xrightarrow{g(W)} \mathcal{F}(W)$$

⊛ Hom does not commute with taking stalks.

$\mathcal{H}om(\mathcal{F},\mathcal{G})_p$ not isomorphic to $Hom(\mathcal{F}_p,\mathcal{G}_p)$. But there is at least a map from one of these to the other.

- $\mathcal{H}om(\mathcal{F},\mathcal{G})_p$ consists of $\{(f, \text{open } U),\ p\in U,\ f\in \mathcal{H}om(\mathcal{F},\mathcal{G})(U) = Mor(\mathcal{F}|_U,\mathcal{G}|_U)\}$ up to equivalence relation
- $Hom(\mathcal{F}_p,\mathcal{G}_p)$: $f:\mathcal{F}_p\to\mathcal{G}_p$ just a map between sets s.t. equivalence relation makes it well-defined map.

One can see that there is map $\mathcal{H}om(\mathcal{F},\mathcal{G})_p \to Hom(\mathcal{F}|_p, \mathcal{G}|_p)$ sends

- $(f, U) \in \mathcal{H}om(\mathcal{F}, \mathcal{G})_P$, which is just morphism of sheaves $f: \mathcal{F}|_U \to \mathcal{G}|_U$ mean (by 2.3A) it induces morphism of stalk $f_P: \mathcal{F}|_P \to \mathcal{G}|_P$.

- Check that equivalence relation makes this map well-defined.

2.3.4 If $\mathcal{F}, \mathcal{G}$ sheaves of abelian groups on X, $\mathcal{H}om_{Ab_X}(\mathcal{F}, \mathcal{G})$ defined as $\mathcal{H}om(\mathcal{F}, \mathcal{G})(U)$ to be maps between sheaves of abelian groups $\mathcal{F}|_U \to \mathcal{G}|_U$.

$\mathcal{H}om_{Ab_X}(\mathcal{F}, \mathcal{G})$ has natural structure of sheaf of abelian groups.

i.e. $\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = Mor(\mathcal{F}|_U, \mathcal{G}|_U)$ is abelian group with addition: $f+g: \mathcal{F}|_U \to \mathcal{G}|_U$

$$(f+g)(W): \mathcal{F}(W) \to \mathcal{G}(W) \text{ defined as } (f+g)(W)\, w = f(W)\, w + g(W) w$$

since $\mathcal{G}(W)$ is abelian.

Call $\mathcal{H}om_{Mod\, \mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ the dual of $\mathcal{O}_X$-module $\mathcal{F}$. and denote $\mathcal{F}^\vee$.

2.3D (an important exercise)

(a) If $\mathcal{F}$ sheaf on X, then $\mathcal{H}om(\underline{\{p\}}, \mathcal{F}) \cong \mathcal{F}$ where $\underline{\{p\}}$ is constant sheaf associated to one element set $\{p\}$.

Isomorphism between sheaf

i.e. $\mathcal{H}om(\underline{\{p\}}, \mathcal{F})(U) \cong \mathcal{F}(U)$ as sets

$Mor(\underline{\{p\}}|_U, \mathcal{F}|_U) \cong \mathcal{F}(U)$

Morphism between sheaves

$f \in Mor(\underline{\{p\}}|_U, \mathcal{F}|_U)$

has $f(W) : \underline{\{p\}}(W) \to \mathcal{F}(W)$

$\left\{ \begin{array}{l} \text{continuous} \\ \text{map } W \to \{p\} \end{array} \right\} \to \mathcal{F}(W)$

just 1 element

$f \in Mor(\underline{\{p\}}|_U, \mathcal{F}|_U)$ maps to $f(U)(1) = x$

where $f(U) : \{1 : U \to \{p\}\} \to \mathcal{F}(U)$

Inverse: from $\mathcal{F}(U) \to \mathcal{H}om(\underline{\{p\}}, \mathcal{F})(U)$

$x \mapsto f$ where $f(W) : \{\text{map } W \to \{p\}\} \to \mathcal{F}(W)$

that sends $1 \mapsto res_{U,W}\, x$

In particular, $f(U) : 1 \mapsto x$

2.3.5 Presheaves of abelian groups ("presheaf $\mathcal{O}_X$-modules") form abelian category

Consider (pre)sheaves of abelian groups. One can add maps of presheaves and get another map of presheaves: if $\phi, \psi : \mathcal{F} \to \mathcal{G}$ then define $\phi + \psi$ by $(\phi+\psi)(U) = \phi(U) + \psi(U)$

$\Rightarrow$ (Pre)sheaves of abelian groups form additive category, i.e. Morphisms between 2 presheaves form abelian group; there is 0-object (i.e. both final and initial); and one can take finite products

If $\phi : \mathcal{F} \to \mathcal{G}$ morphism of presheaves, define presheaf kernel $\ker_{pre}\phi$ by $(\ker_{pre}\phi)(U) := \ker\phi(U)$

2.3.E Show that $\ker_{pre}\phi$ is presheaf (of abelian groups)

⊛ Recall: What is kernel of a morphism? $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ group hom

It is the morphism: $f_U : \ker_{pre}\phi(U) \to \mathcal{F}(U)$ s.t. $\phi(U) \circ f_U = 0$ and is universal in this

$$\begin{tikzcd}
A \arrow[d, dashed, "\exists!"'] \arrow[dr] \arrow[drr, bend left, "0"] & & \\
\ker_{pre}\phi(U) \arrow[r, "f_U"'] \arrow[rr, bend right, "0"'] & \mathcal{F}(U) \arrow[r, "\phi(U)"'] & \mathcal{G}(U)
\end{tikzcd}$$

⊛ How to determine restriction map $res^{ker}_{V,U} : Ker_{pre}\phi(V) \to ker_{pre}\phi(U)$?

Chasing following diagram :

$$\begin{tikzcd} 0 \arrow[r] & ker_{pre}\phi(V) \arrow[r, "f_V"] \arrow[d, dashed, "\exists!"] & \mathcal{F}(V) \arrow[r, "\phi(V)"] \arrow[d, "res^{\mathcal{F}}_{V,U}"] & \mathcal{G}(V) \arrow[d, "res^{\mathcal{G}}_{V,U}"] \\ 0 \arrow[r] & ker_{pre}\phi(U) \arrow[r, "f_U"] & \mathcal{F}(U) \arrow[r, "\phi(U)"] & \mathcal{G}(U) \end{tikzcd}$$

- Group hom map

- Existence : $ker_{pre}\phi(V) \xrightarrow{\quad} \cdot \xrightarrow{\quad} \cdot \downarrow \mathcal{G}(U)$ is zero map and commutes with

$$\begin{tikzcd} ker_{pre}\phi(V) \arrow[d] \arrow[dr] & \\ \mathcal{F}(U) \arrow[r] & \mathcal{G}(U) \end{tikzcd}$$

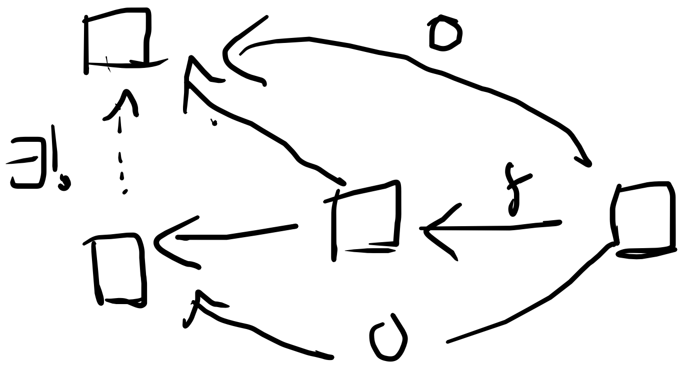
so by universal property of kernel of $\phi(U)$, there exists restriction map. s.t. above diagram commutes.

⊛ Check $res^{ker}_{U,U} = id_{(ker_{pre}\phi)(U)}$ because identity map also satisfies the comm diagram.

Check : $res^{ker}_{U,V} \circ res^{ker}_{W,U} = res^{ker}_{W,V}$ $\Rightarrow$ essentially stack two comm diag on top of each other.

⊛ Presheaf cokernel $\operatorname{coker}_{pre}\phi$: $(\operatorname{coker}_{pre}\phi)(U) := \operatorname{coker}\phi(U)$.

- Recall: coker of $f: U \to V$ is hom $f': V \to \operatorname{coker} f$ s.t. $f' \circ f = 0$ and universal with respect to this property.



- Restriction map $\operatorname{res}^{\operatorname{coker}}_{U,V} : (\operatorname{coker}_{pre}\phi)(U) \longrightarrow (\operatorname{coker}_{pre}\phi)(V)$

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\phi(U)"] \arrow[d, "\operatorname{res}^{\mathcal{F}}_{U,V}"'] & \mathcal{G}(U) \arrow[r] \arrow[d, "\operatorname{res}^{\mathcal{G}}_{U,V}"] & (\operatorname{coker}_{pre}\phi)(U) \arrow[r] \arrow[d, dashed, "\exists!"] & 0 \\
\mathcal{F}(V) \arrow[r, "\phi(V)"] & \mathcal{G}(V) \arrow[r] & (\operatorname{coker}_{pre}\phi)(V) \arrow[r] & 0
\end{tikzcd}$$

2.3.F Presheaf cokernel satisfies universal property of cokernel in cat of presheaves

$$\begin{tikzcd}
\mathcal{H} & & \\
\operatorname{coker}_{pre}\phi \arrow[u, dashed, "\exists!"] & \mathcal{G} \arrow[l] \arrow[ul] & \mathcal{F} \arrow[l, "\phi"'] \arrow[ull, bend right, "0"'] \arrow[ll, bend left, "0"]
\end{tikzcd}$$

- Take $U \subset X$ and we find $(\operatorname{coker}_{pre}\phi)(U)$ to be uniquely identified from diagram

- The restriction map of $\operatorname{coker}_{pre}\phi$ must compatible with

$$\begin{tikzcd}
\operatorname{coker}_{pre} & \mathcal{G} \arrow[l] & \mathcal{F} \arrow[l] \arrow[ll, bend left, "0"]
\end{tikzcd}$$

$$\begin{tikzcd}
\operatorname{coker}_{pre}\phi(U) \arrow[d, "res"] & \mathcal{G}(U) \arrow[l] \arrow[d] & \mathcal{F}(U) \arrow[l] \arrow[d] \arrow[ll, bend right, "0"'] \\
\operatorname{coker}\phi(V) & \mathcal{G}(V) \arrow[l] & \mathcal{F}(V) \arrow[l] \arrow[ll, bend left, "0"]
\end{tikzcd}$$

$\Rightarrow$ restriction map is the one defined for $\operatorname{coker}_{pre}\phi$

$\Rightarrow$ cokernel of presheaves is presheaf cokernel

- In general, presheaves of abelian groups form abelian category
$\Rightarrow$ can define "sub presheaf", "image/quotient presheaf", ...

2.3 G
2.3 H

Show if $0 \to \mathcal{F}_1 \xrightarrow{f_1} \mathcal{F}_2 \xrightarrow{f_2} \mathcal{F}_3 \to 0$ exact in cat of presheaves of top space X

iff $0 \to \mathcal{F}_1(U) \xrightarrow{f_1(U)} \mathcal{F}_2(U) \xrightarrow{f_2(U)} \mathcal{F}_3(U) \to 0$ exact $\forall U \subset X$ open.

$\operatorname{im} f_1 = \ker f_2$.

$\Rightarrow (\operatorname{im} f_1)(U) = (\ker f_2)(U)$

$\operatorname{im} f_1(U) = \ker f_2(U)$

$\Rightarrow$ exact at $\mathcal{F}_2(U)$.

2.3.I

Kernels work with presheaves:

Suppose $\phi : \mathcal{F} \to \mathcal{G}$ morphism of sheaves. Show that presheaf kernel $\ker_{pre} \phi$ is a sheaf. and it satisfies the universal property of kernels.

- To show presheaf kernel $\ker_{pre}\phi$ is a sheaf $\quad \phi : \mathcal{F} \to \mathcal{G}$.

⊛ <u>Identity axiom</u>: $f, g \in (\ker_{pre}\phi)(U) = \ker\phi(U)$ $\quad \{U_i\}_{i\in I}$ open cover U

If $res_{U,U_i} f = res_{U,U_j} g$ then $f = g$.

<u>Proof</u>:

$$\begin{tikzcd} 0 \arrow[r] & \ker\phi(U) \arrow[r, "\varphi_U"] \arrow[d, "res^{\ker}_{U,U_i}"] & \mathcal{F}(U) \arrow[r, "\phi(U)"] \arrow[d, "res^{\mathcal{F}}_{U,U_i}"] & \mathcal{G}(U) \arrow[d, "res^{\mathcal{G}}_{U,U_i}"] \\ 0 \arrow[r] & \ker\phi(U_i) \arrow[r, "\varphi_{U_i}"] & \mathcal{F}(U_i) \arrow[r, "\phi(U_i)"'] & \mathcal{G}(U_i) \end{tikzcd}$$

- Note: What does it mean for $f = g$ in $\ker\phi(U)$? It is equivalent to showing $\varphi_U(f) = \varphi_U(g)$ where $\varphi_U : \ker\phi(U) \to \mathcal{F}(U)$.

  As: one can find $W = \{\varphi_U(f) : f \in \ker\phi(U)\}$ and show that it must be $\ker\phi(U)$, i.e. φ_U injective.

- Since $res^{\ker}_{U,U_i} f = res^{\ker}_{U,U_i} g \Rightarrow \varphi_{U_i} \circ res_{U,U_i} f = \varphi_{U_i} \circ res_{U,U_i} g$

From commutative diagram: $\varphi_{U_i} \circ \mathrm{res}^{\ker}_{U,U_i} f = (\mathrm{res}^{\mathcal{F}}_{U,U_i} \circ \varphi_U) f$

$\Rightarrow \mathrm{res}^{\mathcal{F}}_{U,U_i}(\varphi_U f) = \mathrm{res}^{\mathcal{F}}_{U,U_i}(\varphi_U g) \ \forall i \Rightarrow$ as $\mathcal{F}$ is a sheaf $\varphi_U f = \varphi_U g$

$\Rightarrow f = g$ in $\ker\phi(U)$.

⊛ <u>Gluability axiom</u>: $\{U_i\}_{i\in I}$ open cover U; $f_i \in \ker\phi(U_i)$ so $\mathrm{res}_{U_i, U_i\cap U_j} f_i = \mathrm{res}_{U_j, U_i\cap U_j} f_j$ so exists $f \in \ker\phi(U)$ so $\mathrm{res}_{U,U_i} f = f_i \ \forall i$.

<u>Proof</u>:
$$\varphi_{U_i\cap U_j} \mathrm{res}^{\ker}_{U_i, U_i\cap U_j} f_i = \varphi_{U_i\cap U_j} \mathrm{res}^{\ker}_{U_j, U_i\cap U_j} f_j$$
$$\Rightarrow \mathrm{res}^{\mathcal{F}}_{U_i, U_i\cap U_j} \varphi_{U_i} f_i = \mathrm{res}^{\mathcal{F}}_{U_i, U_i\cap U_j} \varphi_{U_j} f_j$$

$\Rightarrow$ as $\mathcal{F}$ is a sheaf, exists $f \in \mathcal{F}(U)$ so $\mathrm{res}^{\mathcal{F}}_{U,U_i} f = \varphi_{U_i} f_i$.

Also from commutative diagram: $(\mathrm{res}^{\mathcal{G}}_{U,U_i} \circ \phi(U)) f = (\phi(U_i) \circ \mathrm{res}^{\mathcal{F}}_{U,U_i}) f$
$= \phi(U_i)(\varphi_{U_i} f_i) = 0$

$\Rightarrow \phi(U) f = 0$ by identity axiom of $\mathcal{G}$.

$\Rightarrow$ Exists $f' \in \ker\phi(U)$ so $\varphi(U) f' = f$.

2.3.5 X be $\mathbb{C}$ with the classical topology, $\underline{\mathbb{Z}}$ constant sheaf on X associated to $\mathbb{Z}$, $\mathcal{O}_X$ sheaf of holomorphic functions, and $\mathcal{F}$ presheaf of functions admitting holomorphic logarithm. Describe an exact sequence of presheaves on X:

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{i} \mathcal{O}_X \xrightarrow{s} \mathcal{F} \longrightarrow 0$$

where $\underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X$ natural inclusion and $\mathcal{O}_X \longrightarrow \mathcal{F}$ given by $f \mapsto \exp(2\pi i f)$.

- Recall:
  - Constant sheaf on X associated to $\mathbb{Z}$: $\underline{\mathbb{Z}}(U) = \{$continuous map $U \to \mathbb{Z}\}$
  - $\mathcal{O}_X$ sheaf of rings $\mathcal{O}_X(U) =$ ring of holomorphic func from $U \to \mathbb{C}$
  - $\mathcal{F}$ holomorphic func $f: U \to \mathbb{Z}$ so that exists g so $f(z) = e^{g(z)}$ $\forall z \in U$
  - The natural inclusion $\underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X$ means $(f: U \to \mathbb{Z}) \mapsto (f: U \to \mathbb{Z} \subset \mathbb{C})$.

  Some links about holomorphic: https://math.mit.edu/~jorloff/18.04/notes/topic7.pdf https://usamo.wordpress.com/2017/02/16/holomorphic-logarithms-and-roots/

Check exactness: • $\ker i = 0$ due to the natural inclusion.

- Show $\mathrm{im}_{pre}\, i = \ker_{pre} s$ recall $\mathrm{im}\, i = \ker(\mathrm{coker}\, i)$

$$0 \to \underline{\mathbb{Z}} \xrightarrow{i} \mathcal{O}_X \xrightarrow{s} \mathcal{F} \to 0$$

  - Describe $\ker_{pre} s$:

$$\ker s(U) \longrightarrow \mathcal{O}_X(U) \xrightarrow{s(U)} \mathcal{F}(U)$$

  - Describe $\mathrm{im}_{pre}\, i$:

$$\underline{\mathbb{Z}}(U) \longrightarrow \mathcal{O}_X(U) \xrightarrow{\mathrm{coker}\, i} \mathrm{coker}\, i(U)$$

then:

\begin{tikzcd}
(\mathrm{im}\, i)(U) \arrow[r] & \mathcal{O}_X(U) \arrow[r] & \mathrm{coker}\, i(U) \\
\underline{\mathbb{Z}}(U) \arrow[u, "!\exists"] \arrow[ur, "i(U)"'] \arrow[urr, "0"'] & &
\end{tikzcd}

$\mathrm{coker}\, i(U)$ is just all $f: U \to \mathbb{C}$ where exists $z \in U$ so $f(z) \notin \mathbb{Z}$ and the zero map.

$f \in \mathcal{O}_X(U)$ $\quad f(z)$

$\exp(2\pi i f) = 1^{\to U \to \{1\}}$ $\Updownarrow$

$\Leftrightarrow f(z) \in \mathbb{Z}$

suffices to show that this both $\mathrm{im}\, i$ and $\ker s = \underline{\mathbb{Z}}(U)$

§2.4

2.4.A Section of a sheaf of sets is determined by its germs, i.e. the natural map
$$\mathcal{F}(U) \longrightarrow \prod_{p \in U} \mathcal{F}_p \quad \text{is injective}$$
Prove this to sheaves of category of "sets with additional structure"

For $s \in \mathcal{F}(U)$, denote $s_p \in \mathcal{F}_p$ germ of s at $p \in U$.

We have map $\mathcal{F}(U) \to \mathcal{F}_p$ sends $s \mapsto s_p$.

This induces, universal prop of product, $\mathcal{F}(U) \to \prod_{p \in U} \mathcal{F}_p$ sending $s \mapsto \prod_{p \in U} s_p$.

✓ Show injectivity: For $s, v \in \mathcal{F}(U)$, if $s_p = v_p$, i.e.
$$\text{res}_{U, W_p} s = \text{res}_{U, W_p} v \text{ for open } W_p \text{ containing } p,\ W_p \subseteq U$$
$\Rightarrow$ By identity axiom, $s = v$. □

2.4.2 Support of a section. $\mathcal{F}$ sheaf (or just separated sheaf) of abelian groups on X, s global section of $\mathcal{F}$. Let support of s, $\text{Supp}(s)$ be
$$\text{Supp}(s) := \{p \in X,\ s_p \neq 0 \text{ in } \mathcal{F}_p\}$$

2.4.B $\text{Supp}(s)$ closed subset of X.

We show $X \setminus \text{Supp}(s)$ is open. Let $p \in X \setminus \text{Supp}(s)$ then $s_p = 0$ in $\mathcal{F}_p$. This means exists $\tilde{s}_p \in \mathcal{F}(U)$ where open $U \ni p$, so $\tilde{s}_p = 0$

For $q \in U$ then let x_q germ of $\tilde{s}_p$ at $q \Rightarrow x_q = 0$ as $\tilde{s}_p = 0$.

This follows $q \notin \text{Supp}(s)$ for all $q \in U \Rightarrow U \subseteq X \setminus \text{Supp}(s) \Rightarrow$ open □

2.4.3 Def. Element $\prod_{p \in U} s_p$ of $\prod_{p \in U} \mathcal{F}_p$ consists of compatible germs if for all $p \in U$, there is
$$(U_p \text{ open in } U,\ \tilde{s}_p \in \mathcal{F}(U_p))$$
for s_p such that germ of $\tilde{s}_p$ at all $q \in U_p$ is s_q.

2.4.C Any choice of compatible germs for sheaf of sets $\mathcal{F}$ over U is the image of a section of $\mathcal{F}$ over U.

Consider compatible germ $\prod_{p\in U} s_p \in \prod_{p\in U} \mathcal{F}_p$ with $(U_p \underset{\text{open}}{\subseteq} U, \tilde{s}_p \in \mathcal{F}(U_p))$.

* Show $\mathrm{res}_{U_p, U_p\cap U_q} \tilde{s}_p = \mathrm{res}_{U_q, U_p\cap U_q} \tilde{s}_q$

If $U_p\cap U_q$ empty, done! If not, pick $r \in U_p \cap U_q$. Since s_r germ of $\tilde{s}_p$ and $\tilde{s}_q$ at r so exists open $W \ni r$ so $\mathrm{res}_{U_p, W}\tilde{s}_p = \mathrm{res}_{U_q,W}\tilde{s}_q$.

We show W can be $U_p\cap U_q$

We have $\mathrm{res}_{U_p\cap U_q, W}(\mathrm{res}_{U_p, U_p\cap U_q}\tilde{s}_p) = \mathrm{res}_{U_p\cap U_q, W}(\mathrm{res}_{U_q, U_p\cap U_q}\tilde{s}_q)$

Since we can choose any $r\in U_p\cap U_q$; W can range over open cover of $U_p\cap U_q$ $\Rightarrow$ By identity axiom we find

$$\mathrm{res}_{U_p, U_p\cap U_q}\tilde{s}_p = \mathrm{res}_{U_q, U_p\cap U_q}\tilde{s}_q$$

* By gluability axiom for $\mathcal{F}$, we find exists $s \in \mathcal{F}(U)$ so
$$\mathrm{res}_{U, U_q} s = \tilde{s}_q.$$

2.4D) Morphisms are determined by stalks: If ϕ_1 and ϕ_2 morphisms from presheaf of sets $\mathcal{F}$ to sheaf of sets $\mathcal{G}$ that induces same map on each stalk, show that $\phi_1 = \phi_2$.

Given $\phi_{1,p} = \phi_{2,p}$ for all $p \in U \subseteq X$. Show $\phi_1(U) = \phi_2(U)$.

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[dr, shift left, "\phi_2(U)"] \arrow[dr, shift right, "\phi_1(U)"'] & \prod_{p\in U}\mathcal{F}_p \arrow[r] & \mathcal{F}_p \arrow[r, shift left, "\phi_{1,p}"] \arrow[r, shift right, "\phi_{2,p}"'] & \mathcal{G}_p \\
 & \mathcal{G}(U) \arrow[r, hook] & \prod_{p\in U}\mathcal{G}_p \arrow[ur] &
\end{tikzcd}$$

Consider maps $\mathcal{F}(U) \to \cdots \to \mathcal{G}_p$ as above

By universal prop of product, this induces map $\mathcal{F}(U) \to \prod_{p\in U}\mathcal{G}_p$

But as $\phi_{1,p} = \phi_{2,p}$ so both maps

$$\mathcal{F}(U) \underset{\phi_2(U)}{\overset{\phi_1(U)}{\longrightarrow}} \mathcal{G}(U) \hookrightarrow \prod_{p\in U}\mathcal{G}_p$$

Commutes with the diagram so they must be equal.

Since $\mathcal{G}$ sheaf so $\mathcal{G}(U) \hookrightarrow \prod_{p\in U}\mathcal{G}_p$ injective

$\Rightarrow \phi_1(U) = \phi_2(U)$ $\square$