1 Functions between sets

1.1 Injections, surjections, bijections

Different way to see injections, surjections, bijections. One is our usual way: Function $f : A \to B$ is injective if $a \neq a' \implies f(a) \neq f(a')$; f is surjective if for all $b \in B$, there exists $a \in A$ so $f(a) = b$ (or in other words, $\operatorname{im} f = B$. Injections are often drawn $\hookrightarrow$, surjections are often drawn $\twoheadrightarrow$. If f is both injective and surjective then f is bijective.

The second viewpoint: For $A \neq \varnothing$ then f is injective when f has left-inverse (i.e. exists $g : B \to A$ so $g \circ f = \mathrm{id}_A$, f is injective when f has right-inverse (i.e. exists $g : B \to A$ so $f \circ g = \mathrm{id}_B$). Thus, f is bijection iff it has a two-sided inverse.

> The previous definition of injective and surjective maps relied crucially on working directly with the *elements* of our sets. However, the second viewpoint shows that in fact these properties are detected by the way functions are 'organized' among sets. Even if we did not know what 'elements' means, still we could make sense of the notions of injectivity and surjectivity (and hence of isomorphisms of sets) by exclusively referring to properties of functions.

Third viewpoint: Function f is a *monomorphism* if for all sets Z and all functions $\alpha, \alpha' : Z \to A$ if $f \circ \alpha = f \circ \alpha'$ then $\alpha = \alpha'$. One can show f is injective iff it is a monomorphism. Similar thing can be said for *epimorphism* and surjection.

1.2 Canonical decomposition

The reason why we focus our attention on injective and surjective maps is that they provide the basic 'bricks' out of which *any* function may be constructed.

Observe every function $f : A \to B$ determines an equivalence relation $\sim$ on A: for all $a, a' \in A$ then $a \sim a' \iff f(a) = f(a')$.

Theorem 1.2.1. Let $f : A \to B$ be any function, and define $\sim$ as above. Then f decomposes as follows:

$$\begin{tikzcd} A \arrow[r, two heads] \arrow[rrr, bend left, "f"] & (A/\sim) \arrow[r, "\sim", "\tilde{f}"'] & \operatorname{im} f \arrow[r, hook] & B \end{tikzcd}$$

where the first function is the canonical projection $A \to A/\sim$ (mapping elements to its equivalence class), the third function is the inclusion $\operatorname{im} f \subseteq B$, and the bijection $\tilde{f}$ in the middle is defined as $\tilde{f}([a]_\sim) := f(a)$ for all $a \in A$.

Remark 1.2.2 (From the book)**.** While its proof is trivial, this is a result of some importance, since it is the prototype of a situation that will occur several times in this book. It will resurface every now and then, with names such as 'the first isomorphism theorem'.

2 Categories

2.1 Exercises

2.1.1 Aluffi Algebra chapter 0

1. (3.9) For A, B in MSet with equivalence relations $\sim, \sim'$ respectively, recall $A/\sim$ and $B/\sim'$ (definition 1.2.) denote set of equivalence classes of A, B with respect to $\sim$, respectively. We will define morphisms in $\mathrm{Hom}_{\mathsf{MSet}}(A, B)$ as follow: f is a morphism from A to B if $x \sim y$ in A then $f(x) \sim' f(y)$ in B.

   We check that this defines a category MSet. Indeed, identity for each A exists and is the same as identity in Set. Composition for two morphisms is also the same as in Set and the resulting morphism is indeed belongs to MSet. ...

   This category MSet has Set as full subcategory. Indeed, consider sets with equivalence relation $\sim$ such that each equivalence class contains exactly one element in the set. Such sets in MSet are objects in Set.

   Each object in MSet corresponds to an ordinary multiset as defined in §2.2.

2. (3.10) Since the objects of a category C are not (necessarily) sets, it is not clear how to make sense of a notion of 'subobject' in general. In some situations it does make sense to talk about *subobjects*, and the subobjects of any given object A in C are in one-to-one correspondence with the morphisms $A \to \Omega$ for a fixed, special object Ω of C, called a *subobject classifier*.

   Set has subobject classifier $\Omega = \{0, 1\}$. Indeed, subobjects of A in Set is subsets of A. A subset B of A corresponds to morphism $f : A \to \{0, 1\}$ such that $f(B) = 0$.

3 Morphisms

monomorphism,*epimorphism*, *isomorphism*, *automorphism*, *groupoid*.

Remark 3.0.1 (Example 4.10)**.** In Set, a function is an isomorphism iff it is both injective and surjective. But in other categories, this property is not true, i.e. a morphism being both a monomorphism and an epimorphism does not imply that it is an isomorphism.

In Set a function is an epimorphism, that is, surjective, iff it has right-inverse. This may fail in general category such as Grp.

3.1 Exercises

1. (4.1) Consider a choice of composing $f_n \cdots f_1$. If with (and), our choice can be written as $h(gf)$ where h, g, f are compositions of morphisms f_i then by associativity, we can rewrite this as $(hg)f$. Induction on n essentially.

2. (4.2) If such category is a groupoid then the relations must also be symmetric.

3. (4.3) If f has right-inverse g then $f \circ g = \mathrm{id}_B$. For $\alpha, \alpha' \mathrm{Hom}(Z, A)$ where Z an object of category C, if $\alpha \circ f = \alpha' \circ f$ then ...

   The converse does not hold, which can be seen by considering category from Example 3.3, i.e. category from a set endowed with a relation. According to Example 4.10, every morphism in this category is both monomorphism and epimorphism. However, each morphism (a, b) doesn't have to have a right-inverse (b, a).

4. (4.4) One cannot define such subcategory $\mathsf{C}_{\text{nonmono}}$ because f, g being nonmonophism does not imply $f \circ g$ begin nonmonophism.

5. (4.5) We will show that monomorphism in MSet is essentially monomorphism in Set.

   One can first see homomorphisms in MSet as a set-function from $A/\sim$ of object $(A, \sim)$ to $B/\sim'$ of object $(B, \sim')$ where $\sim, \sim'$ are equivalence relations. Therefore, if $f \in \mathrm{Hom}(A, B)$ is a monomorphism in MSet then the corresponding set-function $g : A/\sim \to B/\sim'$ must also be a monomorphism. In other words, if $x, y \in A$ and x is not related to y then $f(x) \neq f(y)$.

   Next, consider object $(Z, \sim'')$ of MSet such that $a \sim'' b$ for all $a, b \in Z$. Consider homomorphisms $\alpha, \alpha' \in \mathrm{Hom}(Z, A)$ such that $f \circ \alpha = f \circ \alpha'$ and that images of α, α' are both subsets of some equivalence class T in A. This brings us back to see α, α' as set-function from Z to T and f as a set-function from T to $f(T)$. If f is an monomorphism so for all such α, α', we obtain $\alpha = \alpha'$. This is equivalent to f as monomorphism on equivalence class T of A in category Set. Since the choice of equivalence class T of A is arbitrary, we obtain that monomorphism in MSet is the same as monomorphism in Set, that is, injective.

   One can also show that epimorphism in MSet is the same as surjection (or epimorphism) in Set.

4 Universal properties

A endowed with B: add extra conditions B to A. For example, $\mathbb{Z}$ endowed with relations $\leq$.
induced?

4.1 Exercises

1. (5.1) Not hard.
2. (5.2) Not hard.
3. (5.3) Similar to proof of Proposition 5.4.
4. (5.4) Initial or Final objects in the category of 'pointed set' is set with exactly one element.
5. (5.5) Obviously function $f : A \to Z$ where Z is any one-element set.
6. (5.6) Call such category C. Consider two arbitrary numbers a, b in $\mathbb{Z}^+$ and if we can show that $\mathsf{C}_{a,b}$ has final objects then that means this category has products.

   An object in $\mathsf{C}_{a,b}$ is the diagram:

$$\begin{tikzcd} & a \\ d \arrow[ur] \arrow[dr] & \\ & b \end{tikzcd}$$

   where $d \mid a, b$. One would guess that the final object for this category is when $d = \gcd(a, b)$ and it is indeed true. Similarly, initial objects for $\mathsf{C}^{a,b}$ is $\mathrm{lcm}(a, b)$ with two natural morphisms from a, b to d.
7. (5.7) Any $A \coprod B$ satisfies a universal property so it is well-defined up to isomorphism.
8. (5.8) Huh ? For any category, $A \times B$ is defined (if exists) to satisfy universal property for product of A and B. How would one define $B \times A$?
9. (5.9) $A \times B \times C$ is final objects in category $\mathsf{C}_{A,B,C}$ containing objects:

$$\begin{tikzcd} & A \\ Z \arrow[ur] \arrow[r] \arrow[dr] & B \\ & C \end{tikzcd}$$

   We will show that $(A \times B) \times C$ also satisfies this universal property. Indeed, for any object Z

and morphisms

$$\begin{tikzcd}
& A \\
Z \arrow[ur, "f_A"] \arrow[r, "f_B"] \arrow[dr, "f_C"'] & B \\
& C
\end{tikzcd}$$

we know there exists a unique morphism $f_{A\times B} : Z \to A \times B$ according to the universal property of $A \times B$ such that the diagram

$$\begin{tikzcd}
& & A \\
Z \arrow[rru, bend left, "f_A"] \arrow[r, "f_{A\times B}"] \arrow[rrd, bend right, "f_B"'] & A \times B \arrow[ru] \arrow[rd] & \\
& & B
\end{tikzcd}$$

commutes. Consider the universal property of $(A \times B) \times C$, we know there exists a unique morphism σ from $Z \to (A \times B) \times C$ such that the diagram

$$\begin{tikzcd}
& & A \times B \\
Z \arrow[rru, bend left, "f_{A\times B}"] \arrow[r, "\sigma"] \arrow[rrd, bend right, "f_C"'] & (A \times B) \times C \arrow[ru] \arrow[rd] & \\
& & C
\end{tikzcd}$$

commutes. From this, we know that σ is the unique morphism from $Z \to (A \times B) \times C$ such that the diagram

$$\begin{tikzcd}
& & A \\
Z \arrow[rru, bend left, "f_A"] \arrow[r, "\sigma"] \arrow[rr, bend right, "f_B"'] \arrow[rrd, bend right, "f_C"'] & (A \times B) \times C \arrow[ru] \arrow[r] \arrow[rd] & B \\
& & C
\end{tikzcd}$$

commutes. Similarly, one can also show $A \times (B \times C)$ satisfies the same universal property, which according to Proposition 5.4, we conclude the two objects are isomorphic in $\mathsf{C}_{A,B,C}$ and according to our definition of morphism in this category $\mathsf{C}_{A\times B\times C}$, we find that two objects $(A \times B) \times C$ and $A \times (B \times C)$ are isomorphic in C.

10. (5.10) Similar to universal property in Exercise 5.9. Such coproducts and products exist in Set, and they are what we expect them to be.

11. (5.11) (a) Relation $\sim$ on $A \times B$ is an equivalence relation so $(A \times B)/\sim$ satisfies the universal property for quotients. Note that $A \times B \to A \to A/\sim_A$ also belongs to this category since

equivalent elements in $A \times B$ have the same image in $A/\sim_A$. Since $(A \times B)/\sim$ is an initial object of this category so there exists a unique function $\sigma_A : (A \times B)/\sim \to A/\sim_A$ such that the following diagram commutes

$$\begin{tikzcd}
(A \times B)/\sim \arrow[rr, "\sigma_A"] & & A/\sim_A \\
& A \times B \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

In particular σ_A sends $\overline{(a_1, b_1)} \mapsto \overline{a_1}$. Similarly, we also find a function $\sigma_B : (A \times B)/\sim \to A/\sim_A$.

(b) To show that $(A \times B)/\sim$ with σ_A, σ_B satisfies the universal property for the product of $A/\sim_A$ and $B/\sim_B$, for any set Z and functions $f_A : Z \to A/\sim_A$ and $f_B : Z \to B/\sim_B$, we need to construct a unique morphism $\sigma : Z \to (A \times B)/\sim$ such that the following diagram commutes

$$\begin{tikzcd}
& & A/\sim_A \\
Z \arrow[r, "\sigma"] \arrow[urr, bend left, "f_A"] \arrow[drr, bend right, "f_B"'] & (A \times B)/\sim \arrow[ur, "\sigma_A"] \arrow[dr, "\sigma_B"'] & \\
& & B/\sim_B
\end{tikzcd}$$

Indeed, for $z \in Z$, pick $a \in A$ so the equivalent class of a in A is $f_A(z)$ and we do similar thing with $b \in B$. With that, we let $\sigma(z) = \overline{(a, b)} \in (A \times B)/\sim$. Note that σ is well-defined and it makes the above diagram commuting. Furthermore, this definition is forced by the commutativity of the diagram so σ is unique.

(c) $(A \times B)/\sim \cong A/\sim_A \times B/\sim_B$ is true from (b) and from the fact that all final objects in a category are isomorphic to each other (Proposition 5.4).

12. (5.12) We find the *fibered products* in $\mathsf{C}_{\alpha,\beta}$, i.e. terminal objects of this category. Let's go to find final object [1] of $\mathsf{C}_{\alpha,\beta}$ first. Our final object is the following commutative diagram

$$\begin{tikzcd}
& A \arrow[dr, "\alpha"] & \\
F \arrow[ur, "f_A"] \arrow[dr, "f_B"'] & & C \\
& B \arrow[ur, "\beta"'] &
\end{tikzcd} \tag{4.1.1}$$ pic:I.5.12_1

such that for any object G in $\mathsf{C}_{\alpha,\beta}$ [2] then there exist a unique morphism $\sigma : G \to F$ such that the following diagram commutes

$$\begin{tikzcd}
& & A \arrow[dr, "\alpha"] & \\
G \arrow[r, "\sigma"] \arrow[urr, bend left, "g_A"] \arrow[drr, bend right, "g_B"'] & F \arrow[ur, "f_A"] \arrow[dr, "f_B"'] & & C \\
& & B \arrow[ur, "\beta"'] &
\end{tikzcd} \tag{4.1.2}$$ pic:I.5.12_2

[1]One would guess this is final object not initial object because category $\mathsf{C}_{A,B}$ has final object and $\mathsf{C}_{\alpha,\beta}$ is a fibered version of $\mathsf{C}_{A \times B}$.

[2]The object should be a commutative diagram but we write this for convenience. Look at the next diagram to spot such object.

Similar universal property can be defined for *fibered coproducts* of $\mathsf{C}^{\alpha,\beta}$. The initial object of this category is a commutative diagram

$$\begin{tikzcd} & A \arrow[dl, "f_A"'] & \\ F & & C \arrow[ul, "\alpha"'] \arrow[dl, "\beta"] \\ & B \arrow[ul, "f_B"] & \end{tikzcd} \tag{4.1.3}$$ pic:I.5.12_3

whose universal property involves following commutative diagram

$$\begin{tikzcd} & & A \arrow[dl, "f_A"'] \arrow[dll, bend right, "g_A"'] & \\ G & F \arrow[l, "\sigma"'] & & C \arrow[ul, "\alpha"'] \arrow[dl, "\beta"] \\ & & B \arrow[ul, "f_B"] \arrow[ull, bend left, "g_B"] & \end{tikzcd} \tag{4.1.4}$$ pic:I.5.12_4

Next, we will go and find fibered product and fibered coproduct in Set. For fibered product in $\mathsf{Set}_{\alpha,\beta}$ then $F = \{(a,b) : a \in A, b \in B, \alpha(a) = \beta(b)\}$ where $f_A : (a,b) \mapsto a$ and $f_B : (a,b) \mapsto b$. Checking this definition then one can see the diagram 4.1.1 does indeed commutes. Next, we try to find morphism σ for any object G as in diagram 4.1.2. For $x \in G$, let $\sigma(G) = (g_A(x), g_B(x))$ then $f_A \circ \sigma = g_A$ and $f_B \circ \sigma = g_B$ so the diagram 4.1.2 commutes. Note σ is unique since it is forced by the commutativity of the diagram.

Next, we find the fibered coproducts of $\mathsf{Set}^{\alpha,\beta}$: In order for the diagram 4.1.3 to be commutative, one should observe:

(a) We want to define f_A on A but we only need to set up conditions for $\alpha(C) \subseteq A$ to achieve a commutative diagram. Hence, for simplicity, one would guess $f_A(A \setminus \alpha(C))$ is a copy of $A \setminus \alpha(C)$ in F that has nothing to do with $f_B(B) \subseteq F$. Similar thing can be said for $B \setminus \beta(C)$.

(b) The only case left is when $C \to \alpha(C) \to F$ and $C \to \beta(B) \to F$. We want these two to match up and ends up at F. It is obvious that one should set up f_A, f_B so $f_A(\alpha(c)) = f_B(\beta(c))$. Note that if we have $c_1, c_2 \in C$ such that $\alpha(c_1) = \alpha(c_2)$ then this forces f_B to $f_B(\beta(c_1)) = f_B(\beta(c_2))$. This suggests us to define an equivalence relation $\sim$ on C such that

$$c_1 \sim c_2 \iff \alpha(c_1) = \alpha(c_2) \text{ or } \beta(c_1) = \beta(c_2).$$

With this, we can let $f_A : \varphi(c) \to [c]_\sim \in C/\sim$ and $f_B : \beta(c) \to [c]_\sim \in C/\sim$.

In general, we choose $F = (C/\sim) \coprod (A \setminus \alpha(C)) \coprod (B \setminus \beta(C))$ and choose f_A, f_B as describe above. This guarantees that 4.1.3 is a commutative diagram.

Next, to show such object satisfies the universal property for fibered coproducts as in diagram 4.1.4, we define σ as follow:

(a) If $[c]_\sim \in (C/\sim) \subseteq F$ then $\sigma([c]_\sim) = g_A(\alpha(c)) = g_B(\alpha(c))$ (the last equal sign is given since G with g_A, g_B is an object of $\mathsf{Set}^{\alpha,\beta}$). This condiiton shows that $g_A = \sigma f_A$ when restricting the domain to $\alpha(C)$ and $g_B = \sigma f_B$ when restricting the domain to $\beta(C)$.

(b) If $a \in A \setminus \alpha(C)$ then $\sigma(a) = g_A(a)$. Similarly, if $b \in B \setminus \alpha(C)$ then $\sigma(b) = g_B(b)$. This guarantees that $(g_A)|_{A \setminus \alpha(C)} = (\sigma f_A)|_{A \setminus \alpha(C)}$ and $g_B = \sigma f_B$ when restricting the domain to $B \setminus \beta(C)$.

In general, σ is defined so $g_A = \sigma f_A$ and $g_B = \sigma f_B$ which implies commutativity of diagram 4.1.4. Furthermore, σ is unique on F since it is defined so $g_A = \sigma f_A$ and $g_B = \sigma f_B$ and we know that one can arrive to any element in F using either f_A or f_B.

Question 4.1.1. Does there exist a category such that it has (finite) products but it doesn't have (finite) coproducts?

References

algchap0 [1] Paolo Aluffi. Algebra: Chapter 0