# BASIC NOTIONS WEBMINAR

# Part I: Homological algebra (Đại số đồng điều)

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## Lecture 1: 31/05/2020

## ⊛ Extension of scalars (Mở rộng hệ số)

ring = commutative ring with identity (vành có đơn vị)

$f: R \to S$ ring hom then $S$ is an R-algebra (R-đại số)

$$\underbrace{\text{R-algebra}}_{\text{R-mod} \quad \text{ring structure}}$$

R-mod:

$R \times S \to S$

$r.s := f(r)s$

If $M$ is R-mod, $S$ is R-algebra. Construct S-mod from $M$ is called extension of scalars.

First we learn about tensor products:

## ⊛ Tensor products (tích tensor)

$M, N$ R-modules

$$M \otimes N := R^{\oplus (M \times N)} \Big/ \left\langle \begin{array}{l} (m, n_1 + n_2) - (m, n_1) - (m, n_2) \\ (m_1 + m_2, n) - (m_1, n) - (m_2, n) \\ (\lambda m, n) - \lambda (m, n) \\ (\lambda m, n) - \lambda (m, n) \\ \lambda \in R;\ m, m_1, m_2 \in M \end{array} \right\rangle$$

$$\sum_{\substack{\text{finite} \\ m \in M \\ n \in N}} \lambda_{m,n} (m, n)$$

An element of $M \otimes N$ is finite sum $\sum_{i=1}^{k} m_i \otimes n_i$

# Lecture : 7/06/2020

Content: Chain complex (phức dây chuyền)
Homology đồng điều
Snake lemma bổ đề rắn
Zigzag lemma
Long exact seq of cohomology
Exactness of Hom, Tensor
Projective / Injective / Flat module Xạ ảnh / Nội xạ / phẳng

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Fix · R = commutative ring with identity

Def. A chain complex (phức dây chuyền) of R-modules is a sequence (finite or infinite)

$(A_\cdot, \partial_\cdot)$ $\cdots \to A_{n+1} \xrightarrow{\partial_n} A_n \xrightarrow{\partial_{n-1}} A_{n+2} \xrightarrow{\partial_{n-2}} \cdots$

s.t. $\partial_{n-1} \circ \partial_n = 0 \ \forall n$ (i.e. $\operatorname{Im} \partial_n \subseteq \ker \partial_{n-1}$)

Vocabulary : $\partial_\cdot$ : differential (vi phân)

$x \in A_n$ : n-chain (dây chuyền bậc n)

Exact at $A_n$ $\Leftrightarrow$ $\operatorname{Im} \partial_n = \ker \partial_{n-1}$

$Z_n(A) := \ker \partial_{n-1} = \{x \in A_n : \partial x = 0\}$ n-cycle chu trình

$B_n(A) := \operatorname{Im} \partial_n = \{x \in A_n : \exists y \in A_{n+1} \text{ so } \partial y = x\}$ n-boundary
n-biên

have $B_n(A_\cdot) \subseteq Z_n(A_\cdot)$

$H_n(A_\cdot) := Z_n(A_\cdot) / B_n(A_\cdot)$ nth homology group of $A_\cdot$
nhóm đồng điều bậc n

· Morphisms between chain complexes: (but are also R-modules)

$$\begin{array}{ccccccccc}
\cdots \to & A_{n+1} & \xrightarrow{\partial} & A_n & \xrightarrow{\partial} & A_{n-1} & \xrightarrow{\partial} & \cdots & \quad A_\cdot \\
& \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & & \\
\cdots \to & B_{n+1} & \xrightarrow{\partial} & B_n & \xrightarrow{\partial} & B_{n-1} & \to & \cdots & \quad B_\cdot
\end{array}$$

A chain map $f_\cdot : A_\cdot \to B_\cdot$ is a collection of R-modules hom
$\{f_n : A_n \to B_n\}$ s.t. $f_n \partial = \partial f_{n+1}$

*Fact: $f_\cdot$ induces canonical homomorphisms of R-modules

$$H_n(f_\cdot) : H_n(A_\cdot) \longrightarrow H_n(B_\cdot)$$

$$H_n(A) = \frac{Z_n(A)}{B_n(A)} \longrightarrow \frac{Z_n(B)}{B_n(B)} = H_n(B_\cdot)$$

- If $x \in Z_n(A)$ i.e. $x \in A_n$, $\partial x = 0$

$\Rightarrow \partial f_n(x) = f_{n-1}(\partial x) = 0 \Rightarrow f_n(x) \in Z_n(B)$

$$\begin{tikzcd} f_n : Z_n(A) \arrow[r] \arrow[d] \arrow[dr, dashed] & Z_n(B) \arrow[d] \\ H_n(A) \arrow[r, "?"] & H_n(B) \end{tikzcd}$$

- If $x \in B_n(A)$ $\exists y \in A_{n+1}$ $\partial y = x$

$\Rightarrow f_n(x) = f_n(\partial y) = \partial f_{n+1} y \in B_n(B)$

$\Rightarrow$ induces map $H_n(A) \to H_n(B)$

$$\begin{tikzcd} \arrow[r] & A_{n+1} \arrow[r, "\partial"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "\partial"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & {} \\ \arrow[r] & B_{n+1} \arrow[r, "\partial"'] & B_n \arrow[r, "\partial"'] & B_{n-1} \arrow[r, "\partial"'] & {} \end{tikzcd}$$

$H_n(f)([x]) = [f_n(x)]$

This is R-mod homomorphism

Furthermore $H_n(f \circ g) = H_n(f) \circ H_n(g)$ ┘

* Cohomology (Đối đồng điều)

$d \circ d = 0$ cochain complex

$$\cdots \longrightarrow A_n \xrightarrow{d} A_{n+1} \xrightarrow{d} A_{n+2} \longrightarrow \cdots$$

$x \in A_n$ : n-cochain đối dây chuyền

$Z_n(A) := \{x \in A_n : dx = 0\}$ n-cocycle (đối chu trình)

$B_n(A) := \{x \in A_n . \ dy = x \ y \in A_{n-1}\}$ n-coboundary (đối biên)

$H_n(A) = Z_n(A) / B_n(A)$ nth cohomology group

Aim: $0 \longrightarrow A_\cdot \xrightarrow{f_\cdot} B_\cdot \xrightarrow{g_\cdot} C_\cdot \longrightarrow 0$

of chain complexes which exact ($\forall n : 0 \to A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \to 0$ exact)

$\overset{?}{\rightsquigarrow}$ Long exact sequence:

$$\begin{tikzcd} H_n(A) \arrow[r, "H_n(f)"] & H_n(B) \arrow[r, "H_n(g)"] \arrow[dl, phantom, ""] & H_n(C) \arrow[dll, "?"'] \\ H_{n-1}(A) \arrow[r, "H_{n-1}(f)"'] & H_{n-1}(B) \arrow[r, "H_{n-1}(g)"'] & H_{n-1}(C) \end{tikzcd}$$

* Snake lemma

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0 \end{tikzcd}$$

coker đối hạch
ker hạch

Consider commutative diagram with exact rows

Then there is exact seq (natural)

$$0 \to \ker\alpha \xrightarrow{f} \ker\beta \xrightarrow{g} \ker\gamma \xrightarrow{\delta} \operatorname{coker}\alpha \xrightarrow{\bar{f'}} \operatorname{coker}\beta \xrightarrow{\bar{g'}} \operatorname{coker}\gamma \to 0$$

- $\ker\alpha \xrightarrow{f} \ker\beta$: If $x \in \ker\alpha$,

  $\alpha x = 0 \Rightarrow \beta f x = 0 = f'(\alpha x)$.

  $\Rightarrow fx \in \ker\beta$

- $\operatorname{coker}\alpha \xrightarrow{\bar{f}} \operatorname{coker}\beta$ : $\forall x \in \alpha(A)$ $\quad f'(\alpha x) = \beta(fx)$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A'/\alpha(A) \arrow[r, dashed, "\bar{f}"] & B'/\beta(B) \end{tikzcd}$$

$\Rightarrow f'(\alpha(A)) \subseteq \beta(B)$

$\Rightarrow \bar{f}: \operatorname{coker}\alpha \to \operatorname{coker}\beta$

$[x'] \mapsto [f'(x')]$

- Exactness at $\ker\beta$: We have $g \circ f = 0$ $\quad \ker g = \operatorname{im} f$

  take $y \in \ker\beta$ so $g(y) = 0 \Rightarrow y = f(x), x \in A$ as exact at $B$

  $\Rightarrow f'(\alpha x) = \beta(fx) = \beta(y) = 0$

  Since $f'$ injective so $\alpha x = 0 \Rightarrow x \in \ker\alpha \Rightarrow y \in f(\ker\alpha)$
  (exact at $A'$)

- Construct $\ker\gamma \xrightarrow{\delta} \operatorname{coker}\alpha$:

  $z \in \ker\gamma \Rightarrow z = g(y), y \in B$ as exact at $C$

  $\Rightarrow \gamma(g(y)) = g'(\beta' y) = 0 \Rightarrow \beta' y \in \ker g' = \operatorname{im} f'$

  $\Rightarrow \beta' y = f'(x')$ for some $x' \in A$ note $x'$ is unique.

  Define $\delta z = [x'] \in \operatorname{coker}\alpha$ $\quad$ 1-1

  Why $[x']$ is unique: If $z = g(y) = g(y_1) \Rightarrow y = y_1 + f(x)$ for $x \in A$
  as $\ker g = \operatorname{im} f$

We have $\beta(y) = f'(x')$ $\quad x, x_1 \in A'$
$\beta(y_1) = f'(x_1')$ $\qquad$ To show $\delta z$ well defined:

need to show $[x'] = [x_1'] \in \operatorname{coker}\alpha = A'/\operatorname{im}\alpha = \alpha(A)$

$\beta(y) = \beta(y_1) + \beta f(x)$

$f'(x') = f'(x_1') + f'\alpha x \Rightarrow x' = x_1' + \alpha x$ as $f'$ injective

$\Rightarrow [x'] = [x_1']$.

- Show $\delta$ hom: take $z_1, z_2 \in C$, $z_1 = g(y_1)$, $z_2 = g(y_2)$

$z_1 + z_2 = g(y_1 + y_2)$

$$\left.\begin{aligned} \beta(y_1) &= f'(x_1') \\ \beta(y_2) &= f'(x_2') \end{aligned}\right\} \Rightarrow \beta(y_1 + y_2) = f'(x_1' + x_2')$$

$\delta(z_1 + z_2) = [x_1' + x_2'] = [x_1'] + [x_2'] = \delta z_1 + \delta z_2$

- Show exactness of $\delta$:
  - $\delta g = 0$: if $y \in \ker\beta$. Show $\delta(g(y)) = 0$

    From def of $\delta$: as $\beta(y) = 0 = f'(0)$ so $\delta(g(y)) = [0] = 0$
  - show $\ker\delta = \operatorname{im} g$: $z \in \ker\gamma$ so $\delta z = 0$

    $z = g(y)$, $\beta(y) = f'(x')$ for some $y \in B$, $x' \in A'$. then $\delta z = [x']$

    as $\delta z = 0$ so $[x'] = 0$ so $x' = 0$ in $A'/\alpha(A) \Rightarrow x' = \alpha(x)$ for some $x \in A$

    $\beta(y) = f'(x') = f'(\alpha x) = \beta(f(x))$

    $\Rightarrow y - f(x) \in \ker\beta \Rightarrow z = g(\underbrace{y - f(x)}_{\in \ker\beta})$ as $g \circ f = 0$.

- Show $\delta$ is "natural":

If following commutes:

$$\begin{tikzcd}
 & A_2 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_2' \arrow[r] & B_2' \arrow[r] & C_2' & \\
 & A_1 \arrow[r] \arrow[d] \arrow[uu] & B_1 \arrow[r] \arrow[d] \arrow[uu] & C_1 \arrow[r] \arrow[d] \arrow[uu] & 0 \\
0 \arrow[r] & A_1' \arrow[r] \arrow[uu] & B_1' \arrow[r] \arrow[uu] & C_1' \arrow[uu] &
\end{tikzcd}$$

(Exercise)

Then:

$$\begin{tikzcd}
\ker\alpha_2 \arrow[r] & \ker\beta_2 \arrow[r] & \ker\gamma_2 \arrow[r, "\delta_2"] & \operatorname{coker}\alpha_2 \arrow[r] & \operatorname{coker}\beta_2 \arrow[r] & \operatorname{coker}\gamma_2 \\
\ker\alpha_1 \arrow[r] \arrow[u] & \ker\beta_1 \arrow[r] \arrow[u] & \ker\gamma_1 \arrow[r, "\delta_1"'] \arrow[u] & \operatorname{coker}\alpha_1 \arrow[r] \arrow[u] & \operatorname{coker}\beta_1 \arrow[r] \arrow[u] & \operatorname{coker}\gamma_1 \arrow[u]
\end{tikzcd}$$

Summary:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
 & \ker\alpha \arrow[r] \arrow[d] & \ker\beta \arrow[r] \arrow[d] & \ker\gamma \arrow[d] \arrow[dddll, bend left, "\delta"] \\
 & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\
 & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[d] \\
 & \operatorname{coker}\alpha \arrow[r] & \operatorname{coker}\beta \arrow[r] & \operatorname{coker}\gamma
\end{tikzcd}$$

⊛ Apply snake lemma to ~~co~~homology:

Given $0 \longrightarrow M_\bullet \xrightarrow{f_\bullet} N_\bullet \xrightarrow{g_\bullet} P_\bullet \longrightarrow 0$ $\quad n \in \mathbb{N}$

Then:

$$\begin{tikzcd}
 & \dfrac{M_{n+1}}{B_{n+1}(M)} \arrow[r, "\overline{f_{n+1}}"] \arrow[d, "\bar\partial"] & \dfrac{N_{n+1}}{B_{n+1}(N)} \arrow[r, "\overline{g_{n+1}}"] \arrow[d, "\bar\partial"] & \dfrac{P_{n+1}}{B_{n+1}(P)} \arrow[r] \arrow[d, "\bar\partial"] & 0 \\
0 \arrow[r] & Z_n(M) \arrow[r, "f_n"] & Z_n(N) \arrow[r, "g_n"] & Z_n(P) &
\end{tikzcd}$$ (⊛)

- $\bar\partial : \frac{M_{n+1}}{B_{n+1}(M)} \to Z_n(M)$ well defined:

  $\partial : M_{n+1} \to M_n$ exact at $M_n \Rightarrow \partial(M_{n+1}) \subseteq Z_n(M)$

  If $x \in B_{n+1}(M) \Rightarrow \partial x = 0 \Rightarrow \partial$ factors through $\frac{M_{n+1}}{B_{n+1}(M)}$

⊛ Apply snake lemma to (⊛):

- $\ker\bar\partial = H_{n+1}(M)$ as $x \in M_{n+1}$: $\bar\partial\bar x = 0 \Leftrightarrow \partial x = 0 \Leftrightarrow x \in Z_{n+1}(M)$

  $\Leftrightarrow \bar x \in Z_{n+1}(M)/B_{n+1}(M) = H_{n+1}(M)$.

- $\operatorname{coker}\bar\partial = H_n(M)$ as

  $\operatorname{coker}\bar\partial = Z_n(M)/\operatorname{im}\bar\partial = Z_n(M)/\operatorname{Im}\partial = H_n(M)$

$$
\begin{tikzcd}
 & H_{n+1}(M) \arrow[r, "H_{n+1}(f)"] \arrow[d] & H_{n+1}(N) \arrow[r, "H_{n+1}(g)"] \arrow[d] & H_{n+1}(P) \arrow[d] \arrow[dddll, bend right, "\delta"'] & \\
 & \dfrac{M_{n+1}}{B_{n+1}(M)} \arrow[r, "\bar{f}_{n+1}"] \arrow[d, "\bar{\partial}"] & \dfrac{N_{n+1}}{B_{n+1}(N)} \arrow[r, "\bar{g}_{n+1}"] \arrow[d, "\bar{\partial}"] & \dfrac{P_{n+1}}{B_{n+1}(P)} \arrow[r] \arrow[d, "\bar{\partial}"] & 0 \\
0 \arrow[r] & Z_n(M) \arrow[r, "f_n"] \arrow[d] & Z_n(N) \arrow[r, "f_n"] \arrow[d] & Z_n(P) \arrow[d] & \\
 & H_n(M) \arrow[r, "H_n(f)"'] & H_n(N) \arrow[r, "H_n(g)"'] & H_n(P) &
\end{tikzcd}
$$

$\Rightarrow$ Long exact

$$
\cdots \to H_{n+1}(M) \to H_{n+1}(N) \to H_{n+1}(P) \to H_n(M) \to H_n(N) \to H_n(P) \to \cdots
$$

$\delta$ is natural since if

$$
\begin{tikzcd}
0 \arrow[r] & M_\bullet \arrow[r] \arrow[d] & N_\bullet \arrow[r] \arrow[d] & P_\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M'_\bullet \arrow[r] & N'_\bullet \arrow[r] & P'_\bullet \arrow[r] & 0
\end{tikzcd}
$$

then we obtain commutative between 2 long exact seq of homologies ...

## ⊛ Chain homotopy: Compare between chain maps (đồng luân)

$$\begin{tikzcd}
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & A_n \arrow[r, "\partial"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dl, "h"'] & A_{n-1} \arrow[r, "\partial"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dl, "h"'] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial"'] & B_n \arrow[r, "\partial"'] & B_{n-1} \arrow[r, "\partial"'] & \cdots
\end{tikzcd}$$

A chain homotopy from $f$ to $g$ is a collection $h: A_n \to B_{n+1}$, $n \in \mathbb{N}$ s.t.

$$f - g = h\partial + \partial h \quad \forall n$$

We say $f$ and $g$ are homotopic if exists such $h$.

Denote $f \sim g$

* **Fact:** Given $A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{i}{\rightrightarrows}} C$. If $f \sim g$ and $i \sim k$ then $if \sim kg$

$$\begin{tikzcd}
\arrow[r] & A_{n+1} \arrow[r] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & A_n \arrow[r] \arrow[d, shift right] \arrow[d, shift left] \arrow[dl, "h"'] \arrow[ddl, dashed] & A_{n-1} \arrow[r] \arrow[d, shift right] \arrow[d, shift left] \arrow[dl, "h"'] \arrow[ddl, dashed] & {} \\
\arrow[r] & B_{n+1} \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & B_n \arrow[r] \arrow[d, shift right] \arrow[d, shift left] \arrow[dl, "h'"'] & B_{n-1} \arrow[r] \arrow[d, shift right] \arrow[d, shift left] \arrow[dl, "h'"'] & {} \\
\arrow[r] & C_{n+1} \arrow[r] & C_n \arrow[r] & C_{n-1} \arrow[r] & {}
\end{tikzcd}$$

• **Proposition:** $f_\bullet, g_\bullet : A_\bullet \to B_\bullet$. If $f \sim g$ then $H_n(f) = H_n(g) : H_n(A) \to H_n(B)$.

**Proof:**

$$\begin{tikzcd}
\arrow[r] & A_{n+1} \arrow[r, "\partial"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & A_n \arrow[r, "\partial"] \arrow[d, shift right] \arrow[d, shift left] \arrow[dl, "h"'] & A_{n-1} \arrow[r, "\partial"] \arrow[d, shift right] \arrow[d, shift left] \arrow[dl, "h"'] & \cdots \\
\arrow[r] & B_{n+1} \arrow[r, "\partial"'] & B_n \arrow[r, "\partial"'] & B_{n-1} \arrow[r, "\partial"'] & \cdot
\end{tikzcd}$$

$f \sim g \Leftrightarrow f - g = h\partial + \partial h$

Recall: $H_n(f)[x] = [f(x)]$ $x \in Z_n(A)$

We have $H_n(f)[x] - H_n(g)[x] = [f(x) - g(x)] = [h\partial x] + [\partial h x] = 0$

as $x \in Z_n(A) \Rightarrow \partial x = 0 \Rightarrow h\partial x = 0$

$\partial(hx) \in B_n(B) \Rightarrow [\partial(hx)] \in \dfrac{Z_n(B)}{B_n(B)}$ is $0$. $\Rightarrow H_n(f) = H_n(g)$ $\square$

- **Def.** $A_\bullet$ and $B_\bullet$ are homotopy equivalence if $\exists f: A_\bullet \to B_\bullet$, $g: B_\bullet \to A_\bullet$
  s.t. $f \circ g \sim id_B$ and $g \circ f \sim id_A$ ⟶ đồng nhất.

- Corollary: If $A_\bullet$ and $B_\bullet$ homotopy equivalence then
  $$H_n(A_\bullet) \underset{H_n(f)}{\simeq} H_n(B_\bullet) \quad \forall n \in \mathbb{N}$$

$$H_n(f \circ g) = H_n(id_B) \Rightarrow H_n(f) \circ H_n(g) = id_{H_n(B)}$$

similar $H_n(g) \circ H_n(f) = id_{H_n(A)}$ $\square$

# Exactness of Hom functors

- **Fact**: $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ of $R$-mods (1)

$M$: $R$-mod

$\mathrm{Hom}_R(M,-): R\text{-mods} \to R\text{-mods}$

Functor hom (hàm tử hom).

$$\begin{tikzcd} M \arrow[r] \arrow[dr, dashed, "f_*"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

$\rightsquigarrow$ $0 \to \mathrm{Hom}_R(M,A) \xrightarrow{f_*} \mathrm{Hom}_R(M,B) \xrightarrow{g_*} \mathrm{Hom}_R(M,C)$ (2)

If (1) is exact then (2) is!

We say $\mathrm{Hom}_R(M,-)$ is left-exact functor.

$f_*$: đẩy xuôi push forward

**Proof**: $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ ← $f$ injective, $gf = 0$, $\ker g \subseteq \mathrm{im} f$

$0 \to \mathrm{Hom}_R(M,A) \xrightarrow{f_*} \mathrm{Hom}_R(M,B) \xrightarrow{g_*} \mathrm{Hom}_R(M,C)$

- $f_*$ injective: If $\varphi: M \to A$ so $f_*\varphi = 0 = f\varphi$. Since $f$ injective $\Rightarrow \varphi = 0$
- If $\varphi: M \to A$ $\quad g_* f_* \varphi = gf\varphi = 0 \Rightarrow g_* f_* = 0$
- $\ker g_* \subseteq \mathrm{im} f_*$: Take $\psi: M \to B$ with $g_*\psi = g\psi = 0$

$\Rightarrow g(\psi(x)) = 0 \ \forall x \in M$

Exact at $B \Rightarrow \psi(x) = f(a_x)$, $a_x \in A$

$a_x$ unique since $f$ injective (exact at $A$)

Define $\varphi: M \to A$ $\quad x \mapsto a_x$

$\Rightarrow \psi(x) = f(\varphi(x)) \Rightarrow \psi = f_*\varphi$

$\varphi$ is a hom as $a_{x+y} = a_x + a_y$ as $f(a_{x+y}) = f(a_x) + f(a_y)$ and $f$ injective (exact at $A$) $\quad \Updownarrow \quad \psi(x+y) = \psi(x) + \psi(y)$ $\square$

Here exact at $B$ is not enough to show exact at $\mathrm{Hom}(M,B)$. Need also exact at $A$.

However, $\mathrm{Hom}_R(M,-)$ is not always right-exact.

• **Def/Prop**: A short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \qquad (1)$$

We have equivalence:

(i) $\exists\, r: B \to A$ s.t. $rf = id_A$ — $f$ has left inverse

(ii) $\exists\, s\ C \to B$ st. $gs = id_C$ — $g$ has right inverse

(iii) $f(A)$ is a direct summand of $B$

($\exists A'$ submodule of $B$ with $B = f(A) \oplus A'$)

We say (1) is a split exact sequence.

Proof: (i) $\to$ (iii) given $r: B \to A$ s.t. $rf = id_A$

We show $B = f(A) \oplus \ker r$.

· $f(A) \cap \ker r = 0$? Let $x \in A$ s.t. $f(x) \in \ker r$ i.e. $rf(x) = 0$

$\Rightarrow x = 0$ as $rf = id_A$

· $f(A) + \ker r = B$? Let $y \in B \Rightarrow r(y) \in A$

$rfr(y) = r(y) \Rightarrow y - \underbrace{fr(y)}_{\in f(A)} \in \ker r$

$\Rightarrow y = \underbrace{y - fr(y)}_{\in \ker r} + \underbrace{fr(y)}_{\in f(A)}$

(ii) $\to$ (iii): Given $s: C \to B$ so $gs = id_C$

We show $B = f(A) \oplus s(C)$.

· $f(A) \cap s(C) = 0$? If $y = f(a) = s(z)$ $a \in A$, $z \in C$, $y \in B$

$0 = gf(a) = gs(z) = z \Rightarrow y = s(0) = 0$.

· $f(A) + s(C) = B$? Take $y \in B \Rightarrow g(y) \in C$

have $gsg(y) = g(y) \Rightarrow sg(y) - y \in \ker g = \operatorname{im} f$

$\Rightarrow sg(y) - y = f(x)$, $x \in A$.

(iii) $\to$ (i) and (ii):

$$0 \to A \underset{r}{\overset{f}{\rightleftarrows}} f(A) \oplus A' \underset{s}{\overset{g}{\rightleftarrows}} C \to 0$$

Each $y$ written as $y = f(x) + y'$ with $x \in A$, $y' \in A'$ unique ($f$ injective)

Define $r(y) := x$

For $r$: $f(x) = f(x) + 0 \Rightarrow r(f(x)) = x$.

R-hom: $\left.\begin{array}{l} y_1 = f(x_1) + y_1' \\ y_2 = f(x_2) + y_2' \end{array}\right\} \Rightarrow y_1 + y_2 = f(x_1 + x_2) + y_1' + y_2'$

$\Rightarrow r(y_1 + y_2) = x_1 + x_2 = r(y_1) + r(y_2)$

For $s$: take $z \in C \Rightarrow z = g(y) = g(f(x) + y') = g(y')$
$y \in B$

Define $s(z) = y'$ $(C \to A')$

- well-define: if $z = g(y_1) = g(y_2)$ with $\begin{array}{l} y_1 = f(x_1) + y_1' \\ y_2 = f(x_2) + y_2' \end{array}$

  $\Rightarrow y_1 - y_2 \in \ker g = f(A)$.

  $\Rightarrow \underbrace{f(x_1 - x_2)}_{\in A} + \underbrace{y_1' - y_2'}_{\in A'} \in f(A) \Rightarrow y_1' = y_2'$

  Show $gs = id_C$: $z \in C$ $\quad gs(z) = g(y') = g(y) = z$
  as $y - y' \in f(A)$.

- Show R-hom: .. $\square$

---

- If $0 \to A \underset{r}{\overset{f}{\rightleftarrows}} B \underset{s}{\overset{g}{\rightleftarrows}} C \to 0$ split exact seq then

$$0 \to \mathrm{Hom}_R(M,A) \underset{r_*}{\rightleftarrows} \mathrm{Hom}_R(M,B) \underset{s_*}{\rightleftarrows} \mathrm{Hom}_R(M,C) \to 0$$

is exact and also split.

Split $\Rightarrow$ exists $r, s \Rightarrow r_*, s_*$

- If $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ right exact $M$ is $R$-mod then

$$0 \to \mathrm{Hom}_R(C,M) \xrightarrow{g_*} \mathrm{Hom}_R(B,M) \xrightarrow{f_*} \mathrm{Hom}_R(A,M)$$ left exact

$\mathrm{Hom}_R(-,M)$ is left-exact functor (hàm phản biến)

# ⊛ Exactness of Tensor functors

Let $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ exact

Then $A \otimes_R M \xrightarrow[f \otimes \mathrm{id}_M]{} B \otimes_R M \xrightarrow[g \otimes \mathrm{id}_M]{} C \otimes_R M \to 0$ also exact.

- We can use Hom-⊗ adjunction
$$\mathrm{Hom}(- \otimes_R M, -) \simeq \mathrm{Hom}(-, \mathrm{Hom}_R(M, -)).$$

- Direct proof:
  - $g \otimes \mathrm{id}_M$ surjective : given $c \otimes m \in C \otimes_R M$
    $c \in C \Rightarrow c = g(b) \Rightarrow (g \otimes_R \mathrm{id}_M)(b \otimes m) = g(b) \otimes m$
    $= c \otimes m$.
  - $(g \otimes_R \mathrm{id}_M)(f \otimes_R \mathrm{id}_M) = (gf \otimes_R \mathrm{id}_M) = 0$
  - $\ker g \otimes_R \mathrm{id}_M \subseteq \mathrm{Im}\, f \otimes_R \mathrm{id}_M$ (harder)
    Let $D := \mathrm{Im}(f \otimes \mathrm{id}_M) \subseteq B \otimes_R M$
    $(g \otimes_R \mathrm{id}_M)|_D = D \Rightarrow \varphi : \frac{B \otimes_R M}{D} \to C \otimes_R M$, $\psi$

- Define $\psi$:
  $C \times M \to \frac{B \otimes_R M}{D}$ ; $(c, m) \mapsto b \otimes_R m \bmod D$
  with $c = g(b)$
  Well-defined as if $c = g(b) = g(b') \Rightarrow b - b' \in \ker g = \mathrm{Im}\, f$
  $\Rightarrow b \otimes m - b' \otimes m \in D$
  induces $C \otimes_R M \to \frac{B \otimes_R M}{D}$ s.t. $\psi(c \otimes m) = b \otimes_R m$
  $g(b) = c$
  i.e. $\varphi : b \otimes m \bmod D \mapsto g(b) \otimes m$
  $\Rightarrow \psi \circ \varphi = \mathrm{id}$ $\varphi \circ \psi = \mathrm{id}$
  $\Rightarrow \frac{B \otimes_R M}{\mathrm{Im}(f \otimes_R \mathrm{id}_M)} \simeq C \otimes_R M \underset{g \otimes_R \mathrm{id}_M}{\simeq} \frac{B \otimes_R M}{\ker(g \otimes_R \mathrm{id}_M)}$
  $\Rightarrow \frac{\ker(g \otimes_R \mathrm{id}_M)}{\mathrm{Im}(f \otimes_R \mathrm{id}_M)} = 0 \Rightarrow \ker = \mathrm{im}$ $\square$

If $\mathrm{Hom}_R(M,-)$ is right-exact $\Rightarrow$ $M$ is projective.

— $\mathrm{Hom}_R(-,M)$ ——— $\Rightarrow$ $M$ is injective.

— $-\otimes_R M$ is left-exact $\Rightarrow$ $M$ is flat.

# Lecture 2: 14/06/2020

Content: Example of (co)homology in
- Simplicial homology (đồng điều đơn hình)
- de Rham cohomology - (đối đồng điều de Rham)

## I. Simplicial homology:

⊕ In $\mathbb{R}^n$, consider $x_0, \dots, x_n$ ($n+1$ points)

they are affinely independent if $x_0 - x_1, \dots, x_0 - x_n$ are linearly independent.

$n=1$ $\quad x_1, x_2$ AI $\iff x_1 \neq x_2$

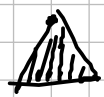
$n=2$ $\quad x_0, x_1, x_2$ AI $\iff x_0, x_1, x_2$ non collinear

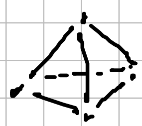
$n=3$ $\quad x_0, x_1, x_2, x_3$ AI $\iff x_0, x_1, x_2, x_3$ non coplanar

⋮

⊕ If that is the case $\left\{ \lambda_0 x_0 + \dots + \lambda_n x_n : \begin{array}{l} \lambda_0, \dots, \lambda_n \geq 0 \\ \lambda_0 + \dots + \lambda_n = 1 \end{array} \right\} = \text{conv}(x_0 \dots x_n)$ tổ hợp lồi của $x_0, \dots, x_n$.

is called $n$-simplex ($n$-đơn hình)

- 0-simplex: 
- 1-simplex: 
- 2-simplex: 

3-simplex  4 2-faces, 6 1-faces, 4 0-faces

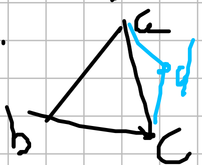
⊕ Take $r+1$ points in $x_0, \dots, x_n$: $x_{i_0}, \dots, x_{i_r}$

$\Rightarrow \{ \lambda_0 x_{i_0} + \dots + \lambda_r x_{i_r} : \lambda_i \geq 0, \lambda_0 + \dots + \lambda_r = 1 \}$ face = diện

is an $r$-simplex, called $r$-face of $\Delta = \text{conv}\{x_0, \dots, x_n\}$

⊕ A simplicial complex in $\mathbb{R}^m$ is a collection $K$ of simplicies s.t. if $\sigma \in K$, then all faces of $\sigma$ in $K$

If $\sigma, \tau \in K$ then $\sigma \cap \tau$ is common face of them

e.g. 

$K = \{ \{abc\}, \{ab\}, \{ac\}, \{b, c\}, \{a\}, \{b\}, \{c\} \}$

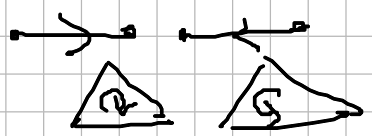
$\{ad\}, \{dc\}, \{d\}, \{adc\}$

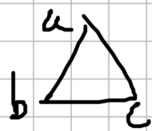
## ⊕ Orientation of simplex : $(r \geq 1)$

We declare that each $r$-simplex has 2 precisely 2 orientation

$r=1$

$r=2$





$(abc)$ denote orientation

- $\sigma \in S_{n+1}$ then $(x_0 \cdots x_n) = (x_{\sigma(0)} \cdots x_{\sigma(n)})$
  $\Leftrightarrow \sigma \in A_{n+1}$ ie. $\operatorname{sgn}(\sigma) = 1$.

If $\operatorname{sgn}(\sigma) = -1$ $\quad (x_0, \dots, x_n) = -(x_{\sigma(0)}, \dots, x_{\sigma(n)})$.

- $K$ oriented simplicial complex (all simplices of dim $\geq 1$ are oriented)

$C_p(K) :=$ free abelian group of $K$ $\qquad C_{-1}(K) := 0$

$$= \left\{ \sum_{j=0}^{n} k_j \sigma_j \quad \sigma_j \in K \text{ is a } p\text{-simplex} \right\}$$

$$\simeq \mathbb{Z}^{\oplus h_p} \qquad h_p = \#\{\text{simplicies of dim } p \text{ of } K\}$$

- Boundary operator: (toán tử biên) $\qquad \partial(a) := 0$

Let $\sigma = (x_0 \cdots x_p)$ be oriented $p$-simplex in $K$ $\qquad a \in C_1(K)$

Define $\partial\sigma := \sum_{i=0}^{p} (-1)^i (x_0 \cdots \hat{x}_i \cdots x_p) \qquad \partial : C_p(K) \to C_{p-1}(K)$

e.g. $a \longrightarrow b$

$\partial(ab) = (b) - (a)$

$\partial(abc) = (bc) - (ac) + (ab)$

$\quad \parallel$

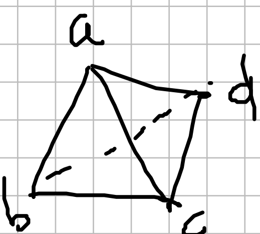
$\partial(bca) = (ca) - (ba) + (bc)$

does not depend on orientation.

$$\partial\partial(abc) = \partial(bc) - \partial(ac) + \partial(ab)$$
$$= (c) - (b) - (c) + (a) + (b) - (a) = 0.$$

Similarly $\partial\partial(x_0 \cdots x_n) = \sum_{i=0}^{n} (-1)^i \, \partial(x_0, \dots, \hat{x}_i, \dots, x_n)$

$$= \sum_{i=0}^{n} \left( \sum_{j<i} + \sum_{j>i} \right) = 0$$



$\partial(abcd) = (bcd) - (acd) + (abd) - (abc)$

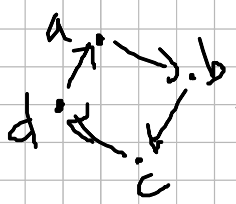
$\partial\partial(abcd) = (cd) - (bd) + (bc) - (cd) + (ad) - (ac)$
$\quad + (bd) - (ad) + (ab) - (bc) + (ac) - (ab)$
$\quad = 0$

We obtain $0 \xleftarrow{\partial} C_0(K) \xleftarrow{\partial} C_1(K) \xleftarrow{\partial} C_2(K) \leftarrow \cdots$

$\partial \circ \partial = 0$

$Z_p(K) = \{c \in C_p(K) \ \partial c = 0\}$ p-cycle

$B_p(K) = \{c \in C_p(K), \ \exists c' \in C_{p+1}(K) \ c' = \partial c\}$ p-bandaries



then $(ab)+(bc)+(cd)+(da)$ is a 1-cycle

$H_p(K) := \dfrac{Z_p(K)}{B_p(K)}$ pth simplicial homology group

(*) $H_0$? $\quad B_0(K) = \{\partial c : c \in C_1(K)\}$

$Z_0(K) = \{(a) : \partial(a) = 0\} = C_0(K)$

$H_0(K) = Z_0(K)/B_0(K) \qquad (x)-(y) \in B_0(K)$

$\Leftrightarrow$ x, y is in the same connected component of $K = \bigcup_{\sigma \in K} \sigma$

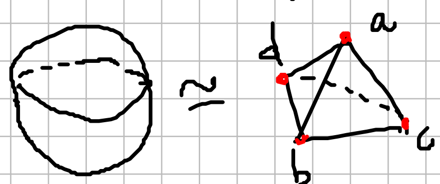
e.g.

$\partial((xz)+(zy)) = (y)-(x)$

$H_0 = \dfrac{C_0(K)}{B_0(K)} = \mathbb{Z}^{\oplus b_0} \quad b_0 = \#\{\text{connected components}\}$

thành phần liên thông.

e.g.



0-simplices $(a),(b),(c),(d)$

1-simplices : $(ab),(bc),(ac),(cd),$
$(da),(bd)$

2-simplices : $(abc),(bcd),(cda),$
$(dab)$

$0 \leftarrow C_0(K) \leftarrow C_1(K) \leftarrow C_2(K) \leftarrow 0 \leftarrow$

• $p=0$: $Z_0(K) = C_0(K)$ since $\partial(a) = \partial(b) = \partial(c) = \partial(d) = 0$.

$x(a)+y(b)+z(c)+t(d) \in B_0(K) \Leftrightarrow x+y+z+t = 0$

as $(b)-(a) = (ab),\ (c)-(b) = (bc),\ (d)-(c) = (dc)$.

$(b) \sim (a)$ in $B_0(K)$ $\quad (c) \sim (b) \quad (d) \sim (c)$ in $B_0(K)$

$H_0(K) = \dfrac{Z_0(K)}{B_0(K)} = \dfrac{\mathbb{Z}(a) \oplus \mathbb{Z}(b) \oplus \mathbb{Z}(c) \oplus \mathbb{Z}(d)}{\langle b-a, c-b, d-c \rangle} \simeq \mathbb{Z}(a) \to$ 1-connected component.

• $\underline{p=1}$: $Z_1(K) = \{ x_1(ab) + x_2(bc) + x_3(cd) + x_4(da) + x_5(ac) + x_6(bd)$
s.t. $x_1 - x_4 + x_5 = x_1 - x_2 - x_6$ ..\}
$= x_2 - x_3 + x_5 = x_3 - x_4 + x_6 = 0$
(*)

$\Rightarrow$

$$\partial(\cdots) = x_1(b-a) + x_2(c-b) + x_3(d-c) + x_4(a-d) + x_5(c-a) + x_6(d-b)$$

$B_1(K) = \langle \partial(abc), \partial(bcd), \partial(abd), \partial(acd) \rangle$

$$y_1 \partial(abc) + y_2 \partial(bcd) + y_3 \partial(abd) + y_4 \partial(acd)$$
$$= x_1(ab) + x_2(bc) + x_3(cd) + x_4(da) + x_5(ac) + x_6(bd)$$

Compare coef: $(ab)$: $x_1 = y_1 + y_3$ $\quad$ $(cd)$ $x_3 = y_2 + y_4$
$(bc)$ $x_2 = y_2 + y_1$ $\quad$ $(da)$ $x_4 = y_3 + y_4$
$(ac)$ $x_5 = y_4 - y_1$ $\quad$ $(bd)$ $x_6 = y_3 - y_2$

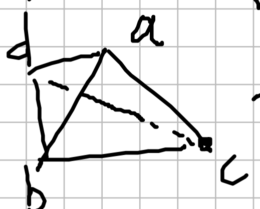
This will imply (*) so $B_1(K) = Z_1(K) \Rightarrow H_1(K) = 0$.

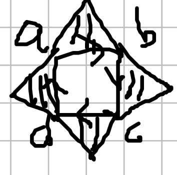
• $\underline{p=2}$ $\quad H_2(K) = \dfrac{Z_2(K)}{B_2(K)}$ $B_2(K) = 0$ since $C_3(K) = 0$
$= Z_2(K)$

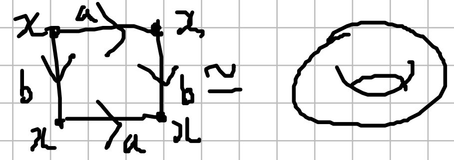
$$x(abc) + y(bcd) + z(cda) + t(dab) \in C_2(K)$$

$$\partial(\cdot) = 0 \Leftrightarrow \begin{cases} (ab): & x + t = 0 \\ (cd) & y + z = 0 \\ (bc) & y + x = 0 \\ (da) & t + z = 0 \\ (ac) & x - z = 0 \\ (bd) & y - t = 0 \end{cases} \Leftrightarrow \begin{cases} y = -x \\ z = x \\ t = -x \end{cases}$$

$$\Rightarrow Z_2(K) = H_2(K) = \mathbb{Z}\langle (abc) - (bcd) + (cda) - (dab) \rangle \simeq \mathbb{Z}.$$

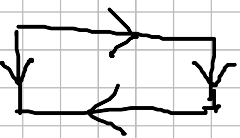
Why $H_2(Z) = \mathbb{Z}$  $\rightarrow \partial(abcd)$ the "hole" that appears in $H_2(K)$.

e.g.  has hole abcd $\rightarrow$
Which will appear in $H_1(K)$

Examples: 


torus = xuyến $\mathbb{Z}/2\mathbb{Z}$


projective space mặt phẳng xạ ảnh



Klein bottle

- **Betti numbers**: A-abelian group

$$\operatorname{rank} A := \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} A)$$

e.g. $\mathbb{Z}^2 \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \Rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} A \simeq \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^2 \simeq \mathbb{Q}^2$.

$\operatorname{rank} A = 2$

Let $\operatorname{rank} H_p(K) = b_p(K)$ Betti numbers of $K$.

$$\chi(K) := b_0 - b_1 + b_2 - \dots = \sum_{i \geq 0} (-1)^i b_i.$$

Euler Poincare characteristic

$n_p = \#\{\text{simplices of dim } p \text{ of } K\}$

$C_p(K) \simeq \mathbb{Z}^{\oplus n_p} \Rightarrow \operatorname{rank} C_p = n_p.$

- Lemma. $\operatorname{Rank} \frac{A}{B} = \operatorname{rank} A - \operatorname{rank} B$

From $0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} C_2 \xleftarrow{\partial} C_3 \leftarrow \cdots$

$\partial: C_p \to C_{p-1} \Rightarrow \frac{C_p}{Z_p} \simeq B_{p-1} \Rightarrow n_p - \operatorname{rank} Z_p = \operatorname{rank} B_{p-1}$

We have $n_0 - n_1 + n_2 - n_3 + \dots$

$$= \operatorname{rank} Z_0 - (\operatorname{rank} Z_1 + \operatorname{rank} B_0) + (\operatorname{rank} Z_2 + \operatorname{rank} B_1) - \dots$$

$$= \operatorname{rank} Z_0 - \operatorname{rank} B_0 - (\operatorname{rank} Z_1 - \operatorname{rank} B_1) + \dots$$

$$= b_0 - b_1 + b_2 \dots = \chi$$

$n_0 = V,\ n_1 = E,\ n_2 = F$ $\quad n_0 - n_1 + n_2$ unchanged under triangulation

For ⊖, we find $b_0 = 1, b_1 = 0, b_2 = 1 \Rightarrow \chi = 2$

$\to V - E + F = 2 \dots$

## II. de Rham cohomology

$V$: $k$-vectorspace $\qquad \langle , \rangle : V \times V^\vee \to k$ $\qquad$ bilinear nondegenerate (ko suy biến)

$(v, \varphi) \mapsto \varphi(v)$

If $\langle , \rangle : V \times W \to k$
$(x, y) \mapsto \langle x, y \rangle$ bilinear and nondegenerate

i.e. $\langle x, y \rangle = 0 \ \forall y \Rightarrow x = 0$

$\langle x, y \rangle = 0 \ \forall x \Rightarrow y = 0$

$\Longrightarrow \varphi : W \to V^\vee$
$y \mapsto \langle \cdot, y \rangle \qquad [x \mapsto \langle x, y \rangle]$

$\langle , \rangle$ being bilinear and nondegenerate $\Rightarrow \varphi$ bijection

- char $k = 0$ $(k = \mathbb{R}, \mathbb{C})$ $\quad \dim V = n$

$\Lambda^p(V) \times \Lambda^p(V^\vee) \to k$

$(x_1 \wedge \cdots \wedge x_p, \varphi_1 \wedge \cdots \wedge \varphi_p) \mapsto \frac{1}{p!} \det[\varphi_i(x_j)]$ → định thức $\Rightarrow$ bilinear

To show nondegerate, need

Lemma: $x_1 \wedge \cdots \wedge x_p = 0 \Leftrightarrow x_1, \ldots, x_p$ linearly dependent

as $x_p = \lambda_1 x_1 + \cdots + \lambda_{p-1} x_{p-1}$, note $x \wedge x = 0$ and $x \wedge y = -y \wedge x$.

Conversely, $x_1, \ldots, x_p$ linearly independent

$\Longrightarrow \varphi_1, \ldots, \varphi_p : V \to k \qquad \varphi_j(x_i) = \delta_{ij}$

$\Rightarrow \langle x_1 \wedge \cdots \wedge x_p, \varphi_1 \wedge \cdots \wedge \varphi_p \rangle = \frac{1}{p!} \det[\varphi_i(x_j)] = \frac{1}{p!} \neq 0$

$\Longrightarrow x_1 \wedge \cdots \wedge x_p \neq 0$. $\qquad \square$

- Show nondenerate: ....

$$\Rightarrow \Lambda^p(V^\vee) \cong \Lambda^p(V)^\vee = \mathrm{Hom}(\Lambda^p(V), k)$$
$$= \mathrm{Alt}(\underbrace{V\times\cdots\times V}_{p}, k)$$
$$(\varphi_1\wedge\cdots\wedge\varphi_p)(x_1,\dots,x_p) = \frac{1}{p!}\det[\varphi_i(x_j)]$$

Recall $\dim \Lambda^p(V) = \binom{n}{p}$ $\dim V = n \Rightarrow \Lambda^n(V) = 1$.

$$f: V\to V \Rightarrow \Lambda^n(f): \Lambda^n(V) \to \Lambda^n(V)$$
$$x_1\wedge\cdots\wedge x_n \mapsto f(x_1)\wedge\cdots\wedge f(x_n)$$

$e_1,\dots,e_n$ basis for $V \Rightarrow e_1\wedge\cdots\wedge e_n$ basis $\Lambda^n(V)$ since $\dim\Lambda^n(V)=1$

$$\Rightarrow \Lambda^n(f)(e_1\wedge\cdots\wedge e_n) = \underset{\underset{\mathrm{tr}\,\Lambda^n(f).}{\parallel}}{\det(f)}(e_1\wedge\cdots\wedge e_n).$$

Similarly, $\mathrm{tr}\,\Lambda^p(f) = \sum$ principal $(p\times p)$-minors of $A$.

---

$U\subseteq\mathbb{R}^n$ open $f: U\to\mathbb{R}^n$ is $\mathcal{C}^\infty$, smooth, ...

if its mixed partial derivatives of all deg exist.

$\mathcal{C}^\infty(U,\mathbb{R})$.

$f: U\to\mathbb{R}^n$ $f = \begin{pmatrix} f_1\\ \vdots \\ f_m\end{pmatrix}$ smooth $\mathcal{C}^\infty$ if $f_i\in\mathcal{C}^\infty(U,\mathbb{R})$

Denote: $\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}$ canonical basis for $\mathbb{R}^n$

$dx_1,\dots,dx_n$ ———— $(\mathbb{R}^n)^\vee$

i.e. $dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}$

We have $\Lambda^p((\mathbb{R}^n)^\vee) \cong \Lambda^p(\mathbb{R}^n)^\vee \cong \mathrm{Alt}(\underbrace{\mathbb{R}^n\times\cdots\times\mathbb{R}^n}_{p}, \mathbb{R})$

$$\cong \mathbb{R}^{\binom{n}{p}}$$

Basis $dx_{i_1}\wedge\cdots\wedge dx_{i_p}$ $1\le i_1<i_2<\cdots<i_p\le n$

$$(e_1,\dots,e_p)\mapsto \frac{1}{p!}\det[\underbrace{x_{i_j}(e_k)}_{}]_{jk}$$

$i_j$ coordinate of $e_k$.

A p-differential form on $U$ is a $\mathcal{C}^\infty$-map
dạng vi phân
$$U \to \Lambda^p(\mathbb{R}^n)^\vee \simeq \mathrm{Alt}((\mathbb{R}^n)^p, \mathbb{R})$$
$$\Omega^p(U) := \mathcal{C}^\infty(U, \Lambda^p(\mathbb{R}^n)^\vee)$$
$$f \in \Omega(U) \iff f = \sum_{1 \le i_1 < i_2 < \dots < i_p \le n} f_{i_1 \dots i_p}\, dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_p}$$
with $f_{i_1 \dots i_p} \in \mathcal{C}^\infty(U, \mathbb{R})$

$d : \Omega^0 \to \Omega^1$ (đạo hàm ngoài)
$$\Omega^0(U) = \mathcal{C}^\infty(U, \mathbb{R}) \qquad df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \quad (1\text{-form})$$
$$d : \Omega^p \to \Omega^{p+1} \quad d(f\, x_{i_1} \wedge \dots \wedge x_{i_p}) := df \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p}$$
$$- d^2 f = \sum_{i=1}^n d\left(\frac{\partial f}{\partial x_i}\right) \wedge dx_i = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j \wedge dx_i$$
$$= \sum_{i<j} \underbrace{\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i}\right)}_{\text{Schwartz theo } 0} dx_i \wedge dx_j = 0.$$

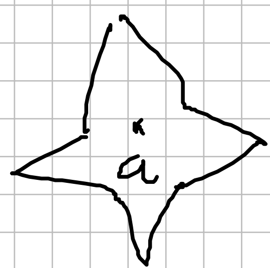
$\Rightarrow$ Cochain complex
$$0 \to \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \to \dots$$

$Z^p(U) = \{w \in \Omega^p(U)\ dw = 0\}$ closed p-forms

$B^p(U) = \{dw : w \in \Omega^p(U)\}$ exact p-forms

$H^p_{dR}(U) := \dfrac{Z^p(U)}{B^p(U)}$ pth de Rham cohomology group of $U$

**Computations:**



Set $U$ is star-shaped if $\exists a \in U$ s.t.
$\forall x \in U, \forall \lambda \in [0,1]\ \lambda a + (1-\lambda) \in U$

$\boxed{\text{Poincare lemma}}$: if $U$ is star-shaped then $H^p(U) = 0\ \forall p \geq 1$

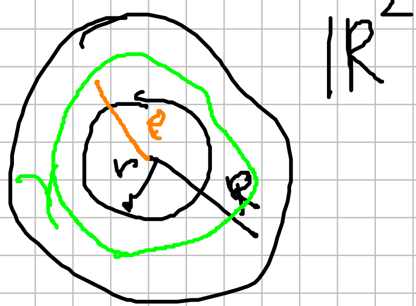
• $p=1$ : $w=\sum_{i=1}^{n} f_i dx_i$ , $dw=0 \Leftrightarrow \frac{\partial f_j}{\partial x_i}=\frac{\partial f_i}{\partial x_j} \ \forall i,j$

Let $a=0$, $f(x)=\int_0^1 \sum_{i=1}^{n} f_i(tx)x_i\,dt$.

$\Rightarrow \frac{\partial f}{\partial x_j}=f_j \quad \forall j=1,\dots,n \quad$ (use $\frac{\partial f_i}{\partial x_j}=\frac{\partial f_j}{\partial x_i}$)

$\Rightarrow df=w$

$\Rightarrow H^1_{dR}(U)=0$.




⊛ If $A$ is vành khăn (annulus)

$$A=\{x\in\mathbb{R}^2 ; r<|x|<R\}$$

$H^1(A)\neq 0$ since with $w=-\frac{y}{x^2+y^2}dx+\frac{x}{x^2+y^2}dy$

then $dw=0$

but $w\notin B^1_{dR}(A)$ : If $w$ was exact

i.e. $\exists f\in \mathcal{C}^\infty(A,\mathbb{R})$ so $\frac{\partial f}{\partial x}=-\frac{y}{x^2+y^2}$, $\frac{\partial f}{\partial y}=\frac{x}{x^2+y^2}$.

Define $\gamma(t)=(\underbrace{\rho\cos t}_{x(t)},\underbrace{\rho\sin t}_{y(t)})$ $t\in[0,2\pi]$

$$\int_\gamma w=\int_0^{2\pi}\frac{-y(t)}{x(t)^2+y(t)^2}x'(t)+\frac{x(t)}{x(t)^2+y(t)^2}y'(t)\,dt$$

$$=\int_0^{2\pi}dt=2\pi.$$

But $\int_\gamma df=\int_0^{2\pi}\frac{\partial f}{\partial x}(\gamma(t))x'(t)+\frac{\partial f}{\partial y}(\gamma(t))y'(t)\,dt$

$$=\int_0^{2\pi}\frac{d}{dt}f(\gamma(t))\,dt=f(\gamma(2\pi))-f(\gamma(0))=0$$

$\Rightarrow w\neq df$, a contradiction.

$w=\frac{-y}{x^2+y^2}dx+\frac{x}{x^2+y^2}dy$.

⊛ $\dim H^1_{dR}(A)=1$

if $\alpha\in Z^1(A)$ then $\alpha-kw$ exact

where $k=\frac{1}{2\pi}\int_\gamma\alpha$ $\Rightarrow H^1_{dR}(A)=\{kw\}$.

Lecture : 25/06/2020

Content : Abelian categories (phạm trù abel)
Derived functors (hàm tử dẫn xuất)

## Linear/Preadditive categories (tiền cộng tính)

**Def.** Category $\mathscr{C}$ is preadditive/linear if

i) $\forall a, b \in \mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(a,b)$ abelian group

ii) $(f+g)h = fh + gh$, $f(g+h) = fg + fh$

(*) We have $A \times B$ "$\cong$" $A \sqcup B$ : (product "$\cong$" coproduct).

$A \xleftarrow{p} A\times B \xrightarrow{q} B$ is coproduct by $\exists$ $A \xrightarrow{i} A\times B$, $B \xrightarrow{j} A\times B$

- $A \xrightarrow{i} A\times B$ is determined by

$$\begin{tikzcd} A \arrow[r, shift left, "i"] \arrow[dr, "1_a"'] & A\times B \arrow[l, shift left, "p"] \arrow[r, "q"] & B \\ & A \arrow[u, dotted] \arrow[ur, "0"'] & \end{tikzcd}$$

i.e. $pi = 1_a$
$qi = 0$

Similarly, $\exists$ $B \xrightarrow{j} A\times B$ so $pj = 0$, $qj = 1_B$

- We show with $i, j$ then $a\times b$ is a coproduct :

$$\begin{tikzcd} a \arrow[r, shift left, "i"] & a\times b \arrow[l, shift left, "p"] \arrow[r, shift right, "q"'] & b \arrow[l, shift right, "j"'] \\ & a\times b \arrow[ul, "p"] \arrow[u, shift left, "ip+jq"] \arrow[u, shift right, "1_{a\times b}"'] \arrow[ur, "q"'] & \end{tikzcd}$$

Consider $ip+jq$

$p(ip+jq) = 1_a p + 0q = p$

$q(ip+jq) = 0p + 1_B q = q$

By universal property of $a\times b \Rightarrow ip + jq = 1_{a\times b}$

Thus, we obtains :

$$\boxed{\begin{aligned} &pi = 1_a,\ qi = 0 \\ &pj = 0,\ qj = 1_B \\ &ip + jq = 1_{a\times b} \end{aligned}} \qquad (*)$$

- Conversely, given $a \xrightarrow{i} a\sqcup b \xleftarrow{j} b$ we can construct $a \xleftarrow{p} a\sqcup b \xrightarrow{q} b$ satisfying the above $(*)$ which follows $a\sqcup b = a\times b$ wrt $p, q$.

- In other direction, given object $c$ and maps $i,j,p,q$

$$\begin{tikzcd} a \arrow[r, shift left, "i"] & c \arrow[l, shift left, "p"] \arrow[r, shift right, "q"'] & b \arrow[l, shift right, "j"'] \end{tikzcd}$$ satisfying (*) then $c = a \times b$ wrt $p,q$
$= a \sqcup b$ wrt $i,j$

- Denote $a \oplus b$ to mean both $a \times b$ and $a \sqcup b$ (biproduct or direct sum)

⊛ **Def: Additive category** $\mathcal{C}$ is
- preadditive
- $a \oplus b$ exists $\forall a, b$ (actually we only need either from above observation $\leftarrow$ $a \times b, p, q$ / $a \sqcup b, i, j$ / $p,q,i,j$ and object $c$)
- has 0 object (i.e. initial and final object)

$0 \xrightarrow{\exists!} a \qquad a \xrightarrow{\exists!} 0$

⊛ **(Co)kernel** can be defined in additive category:
- **Kernel** (hach) of $A \xrightarrow{f} B$ is arrow $K \xrightarrow{i} A$ s.t. $fi = 0$.

that is universal among those who satisfy this property, i.e.
$\forall g: D \to A$ s.t. $fg = 0$
then $\exists! \; j: D \to K$ s.t. $ij = g$

$$\begin{tikzcd} K \arrow[r, "i"] & A \arrow[r, "f"] & B \\ D \arrow[u, "\exists! j"] \arrow[ur, "g"'] & & \end{tikzcd}$$

(In $R$-mod $fg = 0 \Leftrightarrow \operatorname{Img} \subseteq \ker f$)

- **Cokernel** $A \xrightarrow{f} B$ is arrow $B \xrightarrow{p} C$ s.t. $pf = 0$ and $\forall$
$g: B \to D$, $\exists! \; q: D \to C$ s.t. $g = qp$

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "p"] \arrow[dr, "g"'] & C \arrow[d, dashed, "q"] \\ & & D \end{tikzcd}$$

E.g. In $R$-mod, $A \xrightarrow{f} B \xrightarrow{p} B/\operatorname{im} f$

$$\begin{tikzcd} B \arrow[r, two heads, "p"] \arrow[dr, "g"'] & B/\operatorname{im} f \arrow[d, dashed, "q"] \\ & D \end{tikzcd}$$

$gf = 0 \Rightarrow g|_{\operatorname{im} f} = 0$

⊛ **Monomorphism / Epimorphism.**

- $A \xrightarrow{f} B$ is mono if $fx = fy \Rightarrow x = y$ $\quad \forall \begin{tikzcd} C \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & A \end{tikzcd}$

$\Leftrightarrow$ (in preadditive cat) $fz = 0 \to x = 0$

$A \xrightarrow{f} B$ is epic if $xf = 0 \Rightarrow x = 0$ $\forall x : B \to G$

- We have $\ker f \xrightarrow{i} A \xrightarrow{f} B$ then $i$ is monomorphism

  $A \xrightarrow{f} B \xrightarrow{p} \operatorname{coker} B$ then $p$ is epimorphism.

- $f$ is monomorphism $\Leftrightarrow \ker f \cong 0$

  $f$ epic $\Leftrightarrow \operatorname{coker} f \simeq 0$

⊛ Abelian category $\mathcal{C}$ if it is additive and

- every arrow has kernel and cokernel
- If $A \xrightarrow{f} B$ mono, then $A = \ker(B \to \operatorname{coker} f)$

  $A \xrightarrow{f} B$ epic, then $B = \operatorname{coker}(\ker f \to A)$.

- Prop If $\mathcal{C}$ is abelian then $A \xrightarrow{f} B$ isomorphism $\Leftrightarrow$ $f$ mono, epic

Proof $\Leftarrow$ :    $K = \ker f$, $C = \operatorname{coker} f$

$$\begin{tikzcd} K \arrow[r, hook, "i"] & A \arrow[r, "f"] & B \arrow[r, "p"] \arrow[d, equal] & C \\ & & B \arrow[ul, "\exists g"] \arrow[ul, phantom, "1_B", near end] & \end{tikzcd}$$

- $pf = 0 \Rightarrow p = 0$ (since $f$ epic) $\Rightarrow p 1_B = 0$
- But $f : A \to B$ kernel of $B \xrightarrow{p} G \Rightarrow 1_B$ factorises through $f$

  $\Rightarrow \exists g : B \to A$ s.t. $1_B = fg$.
- Similarly, $\exists h : B \to A$ s.t. $hf = 1_A$
- We have $g = 1_A g = (hf)g = h 1_B = h \Rightarrow g = h$

  $\Rightarrow f$ has inverse $g = h$.

⊛ Image of $A \xrightarrow{f} B$ is factorization
$$\begin{tikzcd} A \arrow[r, two heads, "q"] \arrow[dr, "f"'] & I \arrow[d, tail, "e"] \\ & B \end{tikzcd}$$
$q$ epic, $e$ mono

e.g. In $R$-mod $A \twoheadrightarrow \operatorname{Im} f \rightarrowtail B$

We will show image exists and unique up to isomorphism:

want to $I = \ker(B \to \operatorname{coker} f)$.

- <u>Lemma</u>:

$$\begin{tikzcd}
 & A \arrow[dr, "f"] \arrow[ddl, bend right, "g"'] & & \operatorname{coker} m \\
 & & B \arrow[r, "p"] \arrow[ur, bend left, "\phi"] & \operatorname{coker} f \arrow[u, dashed, "\psi"] \\
 I \arrow[urr, bend right, "m"'] & \ker p \arrow[ur, "i"] \arrow[l, dashed, "v"] & &
\end{tikzcd}$$

Given $A \xrightarrow{g} I \xrightarrow{m} B$ with $m$ mono s.t. $f = mg$ then $\exists v : \ker p \to I$ s.t. $i = mv$

<u>Proof lemma</u>: Take $\operatorname{coker} m$ $\phi : B \to \operatorname{coker} m$.

We have $\phi m = 0 \Rightarrow \phi f = \phi m g = 0 \Rightarrow \exists \psi : \operatorname{coker} f \to \operatorname{coker} m$ s.t. $\psi p = \phi \Rightarrow \phi i = \psi p i = 0$ (since $pi = 0$)

$m$ mono $\Rightarrow (I, m)$ is kernel of $\phi$

But $\phi i = 0$ so $\exists v : \ker p \to I$ s.t. $i = mv$ $\square$

- Prove image exists and unique.

<u>Proof</u>:

$$\begin{tikzcd}
K \arrow[dr, "k"] & A \arrow[l, "w"'] \arrow[d, "e"'] \arrow[r, "f"] & B \arrow[r, "p"] & \operatorname{coker} f \\
 & \ker p \arrow[ur, "i"'] \arrow[ul, bend left, "\theta"] \arrow[d, "g"] & & \\
 & T & &
\end{tikzcd}$$

We show $\operatorname{Im} f = \ker p$.

We have $pf = 0$

By def of $\ker p$, $\exists A \xrightarrow{e} \ker p$ s.t. $f = ie$

To show $e$ epic: consider $\ker p \xrightarrow{g} T$ s.t. $ge = 0$.

Let $K \xrightarrow{k} \ker p$ be kernel of $g \Rightarrow \exists A \xrightarrow{w} K$ s.t. $e = kw$

$\Rightarrow ikw = ie = f$

Apply lemma to $f = ikw$ with $ik$ mono, $\exists \theta : \ker p \to \underset{=K}{\ker g}$

$(ik)\theta = i$ since $i$ mono $\Rightarrow k\theta = 1_{\ker g}$

$\Rightarrow g = gk\theta = 0$ (since $gk = 0$)

$\Rightarrow e$ epic

Thus, we obtain $\ker(B \to \operatorname{coker} f)$ is the image.

- Another way:

$$\begin{tikzcd}
\ker f \arrow[r, "i"] & A \arrow[r, "f"] \arrow[d] & B \\
 & \operatorname{coker} i \arrow[ur] &
\end{tikzcd}$$

Can show that $\operatorname{coker}(\ker f \hookrightarrow A)$ also image of $A \xrightarrow{f} B$

i.e. $\operatorname{coker}(\ker) = \ker(\operatorname{coker}) = \operatorname{im} f$.

- Prove uniqueness: Given

$$\begin{tikzcd} A \arrow[r, two heads, "e'"] \arrow[dr, two heads, "e"'] & I' \arrow[r, tail, "i'"] \arrow[d, dashed, leftrightarrow, "\delta"] & B \arrow[r, "p"] & \operatorname{coker} f \\ & \ker p \arrow[ur, "i"'] & & \end{tikzcd}$$

s.t. $i'e' = f$, $e'$ epic, $i'$ mono

Since $pf = 0 = pi'e'$ and $e'$ epic $\Rightarrow pi' = 0$

By def of $\ker p$, $\exists\, \delta : \ker p \to I'$ s.t. $i' = i\delta \Rightarrow \delta$ mono (since $i, i'$ mono)

Also, $\Rightarrow f = i'e' = i\delta e' = ie$. and since $i$ mono

$\Rightarrow \delta e' = e$

Since $e, e'$ epi so $\delta$ epi.

Thus, $\delta$ is isomorphism. $\square$

⊛ If $B \overset{i}{\hookrightarrow} A$ with $i$ mono, call $B$ subobject of $A$

$B \hookrightarrow A \longrightarrow \operatorname{coker}(i) =: A/B$ quotient

⊛ Exact seq: $A \xrightarrow{f} B \xrightarrow{g} C$ is exact at $B$

if $gf = 0$ and $\ker g = \operatorname{im} f$ (as subobject of $B$)

i.e.

$$\begin{tikzcd} \ker g \arrow[r, tail] & B \arrow[r, "g"] & C \\ A \arrow[u, "\operatorname{im} f"] \arrow[ur, "f"'] & & \end{tikzcd}$$

i.e. $A \twoheadrightarrow \ker g$ is epic

- Prop: $0 \longrightarrow A \xrightarrow{f} B$ exact at $A$ $\Leftrightarrow$ $f$ mono

$A \xrightarrow{f} B \to 0$ exact at $B$ $\Leftrightarrow$ $f$ epic

$0 \to A \xrightarrow{f} B \to 0$ exact $\Leftrightarrow$ $f$ isomorphism

$0 \to A \to B \to C$ exact $\Leftrightarrow$ $A = \ker(B \to C)$

$\circledast$ $\mathcal{A}, \mathcal{B}$ abelian categories

Functor $F: \mathcal{A} \to \mathcal{B}$ additive if $\forall f, g : A \to B$ then

$$F(f+g) = F(f) + F(g)$$

- Lemma: $F(0_A) = 0_B$ (use $A \simeq 0 \Leftrightarrow 1_A = 0$)

$F(A \oplus B) \simeq F(A) \oplus F(B)$ $\qquad$ $A \underset{p}{\overset{i}{\rightleftarrows}} A \oplus B \underset{q}{\overset{j}{\leftrightarrows}} B$

- Exact seq $\Rightarrow$ (Co)chain, complex, chain maps, homotopy. can be defined without elements (see previous lecture)

$Hom_{\mathcal{A}}(P,-) : \mathcal{A} \to Ab$ $\quad$ left exact functor.

$Hom_{\mathcal{A}}(-,P) : \mathcal{A}^{op} \to Ab$

- If $Hom_{\mathcal{A}}(P,-) : \mathcal{A} \to Ab$ is exact, i.e. $\forall\ 0 \to A \to B \xrightarrow{f} C \to 0$ exact then $\quad 0 \to Hom_{\mathcal{A}}(P,A) \to Hom_{\mathcal{A}}(P,B) \xrightarrow{f_*} Hom_{\mathcal{B}}(P,C) \to 0$

We say $P$ is projective element, (vật xạ ảnh)

that is, given $f : B \to C$ epic, $\forall g : P \to C$ lifts into $h : P \to B$

$$\begin{tikzcd} & B \arrow[d, "f"] \\ P \arrow[ur, dotted, "\exists h"] \arrow[r, "g"'] & C \end{tikzcd}$$

i.e. $f_*(h) = g = fh$.

$\circledast$ $F : \mathcal{A} \to Ab$ is left-exact if $0 \to A \to B \to C$ exact

$\Rightarrow$ $0 \to F(A) \to F(B) \to F(C)$ exact

- Category $\mathcal{A}$ is said to has enough projectives (đủ xạ ảnh) if $\forall A$, $\exists P$ projective and epi $P \twoheadrightarrow A$.

- $A \in ob(\mathcal{C})$. A projective resolution (giải xạ ảnh) of $A$ is an exact seq $\quad 0 \leftarrow A \leftarrow P_0 \leftarrow P_1 \leftarrow \dots$ s.t. $P_n$ is projective $\forall n \in \mathbb{N}$

**Lemma.** If $\mathcal{A}$ has enough projectives then $\forall$ object $A$ has a projective resolution $A \leftarrow P_*$.

**Proof:** $\exists P_0$ s.t. $0 \leftarrow A \twoheadleftarrow P_0$ $P_0$ projective

$$\begin{tikzcd} 0 & A \arrow[l] & P_0 \arrow[l, two heads, "\varepsilon"'] & & P_1 \arrow[ll, "d_0"'] \arrow[dl, two heads] & & P_2 \arrow[ll] \arrow[dl, two heads] \\ & & \ker\varepsilon \arrow[u, hook] & & \ker d_0 \arrow[u, hook] & & \end{tikzcd}$$

Take projective $P_1$ so $\ker\varepsilon \twoheadleftarrow P_1$

Let $d_0 : P_1 \to P_0$ as in diagram

Define $P_2, \ldots$ similarly

Show exactness at $P_0$, $\ker\varepsilon = \operatorname{im} d_0$ by definition of $P_1$ $\square$

**Lemma:** Give $A \xrightarrow{f} A'$ and two resolutions $A \leftarrow P_*$ projective

$$\begin{tikzcd} 0 & A \arrow[l] \arrow[d, "f"] & P_0 \arrow[l, "\varepsilon"'] \arrow[d, "f_0"] & P_1 \arrow[l, "d_0"'] \arrow[d, "f_1"] & \arrow[l, "d_1"'] \\ 0 & A' \arrow[l] & B_0 \arrow[l, "\varepsilon'"] & B_1 \arrow[l, "d_0'"] & \arrow[l, "d_1'"] \end{tikzcd}$$

$\exists!\ f_\bullet : P_\bullet \to B_\bullet$

$f_\bullet$ "covers" $f$

Moreover, $f_\bullet$ is unique upto a homotopy. $\dashv$

Exact at $A' \Rightarrow \varepsilon'$ epi, and since $P_0$ projective

$\Rightarrow \exists f_0 : P_0 \to B_0$ s.t. $\varepsilon' f_0 = f\varepsilon$

Induction, given

$$\begin{tikzcd} & P_n \arrow[l, "d_n"'] \arrow[d, "f_n"] & \ker d_n \arrow[l, "i_n"'] \arrow[d, "F_n"] & P_{n+1} \arrow[l, two heads, "e_n"'] \arrow[d, "f_{n+1}"] \\ & B_n \arrow[l, "d_n'"] & \ker d_n' \arrow[l, "i_n'"] & B_{n+1} \arrow[l, two heads, "e_n'"] \end{tikzcd}$$

where $i_n e_n = d_{n+1}$

$i_n' e_n' = d_{n+1}'$

We have $d_n' f_n i_n e_n = d_n' f_n d_{n+1} = f_{n-1} \underbrace{d_n d_{n+1}} = 0$

$\Rightarrow d_n' f_n i_n = 0$ ($e_n$ epic) $\Rightarrow \exists F_n : \ker d_n \to \ker d_n'$

s.t. $i_n' F_n = f_n i_n$.

Since $e_n'$ epic and $P_{n+1}$ projective $\Rightarrow \exists f_{n+1} : P_{n+1} \to B_{n+1}$ s.t.

....

- Prove unique upto homotopy:

$$\begin{tikzcd}
0 & A \arrow[l] & P_0 \arrow[l] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "g_0"] \arrow[dr, "h"] & P_1 \arrow[l, "d_0"'] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "g_1"] \arrow[dr, "h"] & \arrow[l, "d_1"'] {} \\
0 & A \arrow[l] \arrow[from=u, shift right, "f"'] \arrow[from=u, shift left, "g"] & B_0 \arrow[l] & B_1 \arrow[l, "d_0'"] & \arrow[l, "d_1'"] {}
\end{tikzcd}$$

Need construct: $h: P_n \to B_{n+1}$ s.t.

$$f_0 - g_0 = d_0' h, \quad f_1 - g_1 = h d_0 + d_1' h, \ ..$$

Since we are in abelian category, can assume $f = 0$, $g_0 = g_1 = \dots = 0$. We have to prove $f_0 \sim 0$.

We have

$$\begin{tikzcd}
A \arrow[d, "0"'] & P_0 \arrow[l, "\varepsilon"'] \arrow[d, "f_0"] \arrow[rd, "g"] \arrow[rrd, "h"] & & \\
A' & B_0 \arrow[l, two heads, "\varepsilon'"] & \ker\varepsilon' \arrow[l, tail, "i"] & B_1 \arrow[l, two heads, "e"]
\end{tikzcd}$$

- $\varepsilon' f_0 = 0\varepsilon = 0 \Rightarrow$ by def of $\ker\varepsilon'$, $\exists g: P_0 \to \ker\varepsilon'$ s.t. $f_0 = ig$
- $P_0$ projective, $e$ epic $\Rightarrow \exists h: P_0 \to B_1$ s.t. $g = eh$.

$$\Rightarrow f_0 = ig = ieh = d_0' h.$$

Induction:

$$\begin{tikzcd}
P_{n-1} \arrow[d, "f_{n-1}"'] \arrow[rd, "h"] & P_n \arrow[l, "d_{n-1}"'] \arrow[d, "f_n"] \arrow[rd, "g"] \arrow[rrd, "h"] & & \\
B_{n-1} & B_n \arrow[l, "d_{n-1}'"] & \ker d_{n-1}' \arrow[l, tail, "i"] & B_{n+1} \arrow[l, two heads, "e"] \arrow[ll, bend left, "d_n'"]
\end{tikzcd}$$

- Show

$$d_{n-1}' f_n = f_{n-1} d_{n-1} = (d_{n-1}' h + h d_{n-2}) d_{n-1} = d_{n-1}' h d_{n-1}$$

$$\Rightarrow d_{n-1}' (f_n - h d_{n-1}) = 0$$

- By def of $\ker d'_{n-1} \Rightarrow \exists g: P_n \to \ker d'_{n-1}$
  s.t. $ig = f_n - hd_{n-1}$
- $P_n$ projective, $e: B_{n+1} \to \ker d'_{n-1}$ epic
  $\Rightarrow \exists\ h: P_n \to B_{n+1}$ s.t. $g = eh$

$\Rightarrow \quad f_n - hd_{n-1} = ig = ief_{n+1} = d'_n h$

$\Rightarrow \quad f_n = hd_{n-1} + d'_n h.$ $\square$

$\circledast$ $\mathcal{A}$ enough projective. $F: \mathcal{A} \to Ab$ right exact

Take any projective resolution of $A$

$$\begin{array}{ccccccccc} 0 & \leftarrow & A & \leftarrow & P_0 & \leftarrow & P_1 & \leftarrow & \cdots \quad \mathcal{A} \\ F \Downarrow \\ 0 & \leftarrow & F(P_0) & \leftarrow & F(P_1) & \leftarrow & F(P_2) & \leftarrow & \cdots \quad Ab \end{array}$$

complex of abelian (no more exact)

Hence, can compute $H^i(F(P_\cdot)) =: L_i F(A)$

ith left derived functor of $F$

- $L_i F(A)$ does not depend choice of projective resolution $A \leftarrow P_*$.

Proof:

$$\begin{array}{ccc} A & \leftarrow & P_\cdot \\ \| & & \downarrow f_\cdot \\ A & \leftarrow & Q_\cdot \\ \| & & \downarrow g_\cdot \\ A & \leftarrow & P_\cdot \end{array}$$

$f_\cdot$ cover $id_A$ from $P_\cdot \to Q_\cdot$

$g_\cdot$ covers $id_A$ from $Q_\cdot \to P_\cdot$

$\Rightarrow (g \circ f)_\cdot$ cover $id_A$ from $P_\cdot \to P_\cdot$

We also have $(1_P)_\cdot \Rightarrow (g \circ f)_\cdot \sim (1_P)_\cdot$

We have $F(P_\cdot) \underset{1_{F(P_\cdot)}}{\overset{F((g \circ f)_\cdot)}{\leftleftarrows}} F(P_\cdot)$

$\Rightarrow H_n(F(g \circ f)) = H_n(1_{F(P_\cdot)})$

$\| \qquad$ similarly $H_n(Ff) \circ H_n(Fg) = 1_{H_n(F(Q_\cdot))}$

$H_n(Fg) \circ H_n(Ff) = 1_{H_n(F(P_\cdot))}$. $\Rightarrow H_n(F(P_\cdot)) \cong H_n(F(Q_\cdot))$ $\lrcorner$

## Functoriality of $L_iF$

$$\begin{tikzcd} 0 & A \arrow[l] \arrow[d] & P_0 \arrow[l] \arrow[d, "f_0"] & P_1 \arrow[l] \arrow[d, "f_1"] & \cdots \arrow[l] \\ 0 & A' \arrow[l] & B_0 \arrow[l] & B_1 \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

$F \rightsquigarrow$

$$\begin{tikzcd} 0 & F(P_0) \arrow[l] \arrow[d, "F(f_0)"] & F(P_1) \arrow[l] \arrow[d, "F(f_1)"] & \cdots \arrow[l] \\ 0 & F(B_0) \arrow[l] & F(B_1) \arrow[l] & \arrow[l] \end{tikzcd}$$

$H_n \rightsquigarrow$ $\quad H_n(Ff_\bullet) : L_nF(A) \to L_nF(B)$

$\quad\quad \| $

$\quad L_nF \quad \Rightarrow L_nF : \mathcal{A} \to Ab$ is a functor.

- $L_0F(A) \simeq F(A)$

$0 \leftarrow A \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots \rightsquigarrow 0 \leftarrow A \leftarrow P_0 \xleftarrow{d_1} P_1$

$F$ right-exact $\Rightarrow$ $0 \leftarrow F(A) \leftarrow F(P_0) \xleftarrow[d_1]{} F(P_1)$ exact

$\Rightarrow F(A) \simeq \operatorname{coker} d_1 = H_0(F(P_\bullet)) = L_0F(A)$ $\square$

- If $A$ projective then $L_n(A) = 0$ $\forall n \geq 1$

since $0 \leftarrow A \xleftarrow{id} A \leftarrow 0 \leftarrow 0 \cdots$

is a projective resolution

- Lemma: $0 \leftarrow A \leftarrow B \leftarrow C \leftarrow 0$ exact with proj resolutions $A \leftarrow P_\bullet$, $C \leftarrow Q_\bullet$

$$\begin{tikzcd} 0 & A \arrow[l] & B \arrow[l] & C \arrow[l] & 0 \arrow[l] \\ 0 & P_\bullet \arrow[l] \arrow[u] & P_\bullet \oplus Q_\bullet \arrow[l] \arrow[u, dashed, "\exists"] & Q_\bullet \arrow[l] \arrow[u] & 0 \arrow[l] \end{tikzcd}$$

then $\exists\, B \leftarrow P_\bullet \oplus Q_\bullet$ s.t. above commutes

Proof:

$$\begin{tikzcd} & P \arrow[r, shift left, "i"] \arrow[d, "\alpha"'] \arrow[dr, "\delta"] & P \oplus Q \arrow[l, shift left, "p"'] \arrow[r, shift left, "q"] \arrow[d, "\beta"] & Q \arrow[l, shift left, "j"] \arrow[d, "\gamma"] \arrow[dl, "g\gamma"'] & \\ 0 & A \arrow[l] & B \arrow[l, "f"] & C \arrow[l, "g"] & 0 \arrow[l] \end{tikzcd}$$

Consider projectives $P, Q$ in $A \leftarrow P_\bullet$, $B \leftarrow Q_\bullet$

- Biproduct $P\oplus Q$ is equipped with $P \underset{p}{\overset{i}{\rightleftarrows}} P\oplus Q \underset{q}{\overset{j}{\leftrightarrows}} Q$

- $f$ epi, $P$ projective $\Rightarrow \delta : P \to B$ $f\delta = \alpha$.

  $\Rightarrow \exists \beta : P\oplus Q \to B$ (from $P \xrightarrow{\delta} B$ and $Q \xrightarrow{g\gamma} B$)

  s.t. $\beta = \delta p + g\gamma q$, $\beta i = \delta$, $\beta j = g\gamma$

- show $\beta$ epic : If $x\beta = 0 \Rightarrow \begin{cases} x\beta i = x\delta = 0 \\ x\beta j = xg\gamma = 0 \end{cases}$ (1)

  $\gamma$ epi $\Rightarrow xg = 0$ $\Rightarrow x$ factors through $\operatorname{coker} g = A$

  $\Rightarrow \exists y : D \to A$ s.t. $x = yf$ so with (1)

  $\Rightarrow yf\delta = 0 \Rightarrow y\alpha = 0 \Rightarrow y = 0$ ($\alpha$ epic)

  $\Rightarrow x = yf = 0$

- Can then construct : Construct $P_1, Q_1$ as $\ker\alpha \twoheadleftarrow P_1$, $\ker\gamma \twoheadleftarrow Q_1$

  then use lemma to construct $P_1 \oplus Q_1$

$$\begin{tikzcd}
 & P_1 \arrow[d, two heads] & P_1\oplus Q_1 \arrow[l] \arrow[d, two heads] & Q_1 \arrow[l] \arrow[d, two heads] & \\
 & \ker\alpha \arrow[d, tail] & \ker\beta \arrow[l] \arrow[d, tail] & \ker\gamma \arrow[l] \arrow[d, tail] & \\
0 & P_0 \arrow[l] \arrow[d, two heads, "\alpha"'] & P_0\oplus Q_0 \arrow[l] \arrow[d, two heads, "\beta"] & Q_0 \arrow[l] \arrow[d, two heads, "\gamma"] & 0 \arrow[l] \\
0 & A \arrow[l] & B \arrow[l] & C \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

- We obtain
$$\begin{tikzcd}
0 & P_0 \arrow[l] \arrow[d] & P_0\oplus Q_0 \arrow[l] \arrow[d] & Q_0 \arrow[l] \arrow[d] & 0 \arrow[l] \\
0 & A \arrow[l] & B \arrow[l] & C \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$ exact

$\overset{F}{\rightsquigarrow}$ $0 \leftarrow F(P_0) \leftarrow F(P_0 \oplus Q_0) \leftarrow F(Q_0) \leftarrow 0$ exact split

By zig-zag lemma, we have long exact seq

$$0 \leftarrow L_0F(A) = F(A) \leftarrow F(B) \leftarrow F(C)$$

$$L_1F(A) \leftarrow L_1F(B) \leftarrow L_1F(C) \xleftarrow{\delta} \cdots$$

đồng cấu nối

$\delta$ connect hom

"Measure how far $F$ from being exact by using derived functors".

⊛ Injective object (vật nội xạ) define similar

$$\Rightarrow \begin{tikzcd} I & A \arrow[l, "\forall g"'] \arrow[d, hook, "f"] \\ & B \arrow[ul, dashed, "\exists g"] \end{tikzcd}$$

$I$ injective, $f$ mono $A \to B$ then $\forall g\ I \to A$, $\exists g: I \to B$ ..

- Cat $\mathcal{A}$ has enough injective $\forall A$ has $A \hookrightarrow I$ for some injective $I$.
- $F: \mathcal{A} \to Ab$ left-exact

$$F\Big\{\ 0 \to A \to I^0 \to I^1 \to \cdots$$ injective resolution

$$0 \to F(I^0) \to F(I^1) \to \cdots$$ complex

$$\Rightarrow R^iF(A) := H^i(F(I^\bullet))$$ ith right derived functor

(dẫn xuất phải của hàm tử khớp trái $F$)

- Long-exact seq of derived functors:

$$0 \to FA \to F(B) \to FC \xrightarrow{\delta} R^1F(A) \to R^1F(A) \to R^1F(C) \to \cdots$$

- Next lecture: Compute right derived func of left-exact $Hom_R(A,-)$ called Ext; left derived functor of $A \otimes_R -$, called Tor.

Lecture 3 : 21/06/2020

Content: Projective, Injective, flat module
Hàm tử tích xoắn ($Tor_n$)
Nhóm mở rộng ($Ext^n$)

R commutative ring with 1

Recall $Hom_R(A,-)$ is left exact  $R\text{-mod} \to Ab$  hàm tử thuận biến
$Hom_R(-,A)$ ———  $R\text{-mod}^{op} \to Ab$  hàm tử phản biến

Def: An R-module P is called projective if $Hom_R(P,-)$ exact.

$0 \to A \to B \xrightarrow{f} C \to 0$ exact

$\Leftrightarrow$ $0 \to Hom_R(P,A) \to Hom_R(P,B) \to Hom_R(P,C) \to 0$ exact

$\Leftrightarrow$ $\forall g: P \to C$
$\exists h: P \to B$

$$\begin{tikzcd} & B \arrow[d, "f"] \\ P \arrow[ur, dashed, "\exists h"] \arrow[r, "\forall g"'] & C \end{tikzcd}$$

f epic

Prop. We have equivalence
(i) P projective  (ii) $\forall$ exact seq $0 \to A \to B \to P \to 0$ splits
(iii) P is direct summand of free module
(iv) $\exists x_i \in P$, $f_i \in Hom_R(P,R)$, $i \in I$
s.t. $\forall x \in P$, $f_i(x) = 0$ for all but finitely many I
and $x = \sum f_i(x) x_i$

Proof: (i) $\Rightarrow$ (ii): Given $0 \to A \to B \xrightarrow{g} P \to 0$

$$\begin{tikzcd} & B \arrow[d, "g"] \\ P \arrow[ur, "f"] \arrow[r, equal, "1_P"'] & P \end{tikzcd}$$

Because P proj, $\exists f: P \to B$ so $gf = 1_P$
$\Rightarrow$ split

(ii) $\Rightarrow$ (iii) :

Fact: Every module is quotient of free module
Given A R-module, let $L = R^{(A)} \xrightarrow{f} A$
$\Rightarrow L/\ker f \simeq A$   $(\lambda_a) \mapsto \sum_{a\in A} \lambda_a a$

Consider $0 \to \ker f \to \underset{\text{free}}{L} \xrightarrow{f} P \to 0$

(ii) $\Rightarrow$ this seq splits. $\Rightarrow$ $L = P \oplus \ker f$.

• (iii) $\Rightarrow$ (iv): Write $R^{(I)} = P \oplus Q$ $\qquad \lambda_i \in R$

Let $x \in P \Rightarrow x = \sum_{i \in I} \lambda_i e_i = \sum_{i \in I} f_i(x) e_i \quad \Rightarrow f_i : P \to R$

$f_i$ is $R$-linear and $f_i(x) = 0$ for almost $i \in I$

Write $e_i = x_i + y_i \quad x_i \in P,\ y_i \in Q$

$\Rightarrow x = \sum_{i \in I} f_i(x)(x_i + y_i) \Rightarrow x = \sum_{i \in I} f_i(x) x_i$

• (iv) $\Rightarrow$ (i): Given $\{x_i\}_{i \in I}$ $x_i \in P$, $\{f_i\}_{i \in I}$ $f_i : P \to R$

So $x = \sum_{i \in I} f_i(x) x_i \quad \forall x \in P$

$$\begin{tikzcd} & A \arrow[d, "\psi"] & a_i \\ P \arrow[ur, dashed, "\exists\phi"] \arrow[r, "g"'] & B & \\ x_i & & g(x_i) \end{tikzcd}$$

Given $\psi : A \twoheadrightarrow B$, $g : P \to B$

$g(x_i) = \psi(a_i)$ for some $a_i \in A$ (as $A \xrightarrow{\psi} B$ surjective)

Define $\phi : P \to A$ : $\phi(x) = \sum_{i \in I} f_i(x) a_i \Rightarrow \psi\phi = g$.

**Corollary:** Free modules are projectives

$\bigoplus_{i \in I} P_i$ is projective $\Leftrightarrow$ $\forall i \in I$, $P_i$ is projective

$R$-mod has enough projective ($\forall A$, $\exists\, P \twoheadrightarrow A$, $P$ projective)

**Def:** $R$-module $Q$ is injective if $\mathrm{Hom}_R(-, Q)$ is exact

$0 \to A \to B \to C \to 0$

$\Rightarrow 0 \to \mathrm{Hom}_R(C, Q) \to \mathrm{Hom}_R(B, Q) \to \mathrm{Hom}_R(A, Q) \to 0$

$\Leftrightarrow$ $\forall f : A \to B$ injective
$\forall g : A \to Q$
$\exists h : B \to Q$ s.t. $gf = h$

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, dashed, "g"] & \\ B \arrow[r, dashed, "h"'] & Q \end{tikzcd}$$

• <u>Admit</u>: R-mod has enough injectives

$(\forall A,\ A \hookrightarrow Q,\ Q \text{ injective})$

<u>Baer criterion</u> $Q$ is injective $\iff$ $\forall$ ideal $I$ of $R$

$\forall I \to Q$ extends to $R \to Q$ along $I \subseteq R$

<u>Proof</u>: $\Longrightarrow$ Clear $I \hookrightarrow Q$, $I$ $R$-module

$\Longleftarrow$ Let $A$ submodule $B$, $g: A \to Q$

$$\begin{tikzcd} A \arrow[d, phantom, "\cap"] \arrow[dr, "g"] & \\ B \arrow[r, dashed, "f?"'] & Q \end{tikzcd}$$

- Consider pairs $(C, f)$ with $A \subseteq C \subseteq B$ and $f: C \to Q$ s.t. $f|_A = g$

$(C,f) \leq (C', f')$ if $C \subseteq C'$ and $f'|_C = f$

$\{(C_i, f_i) : i \in I\}$ is totally ordered

$C = \bigcup_{i \in I} C_i$ ; $f: C \to Q$ $\quad x \mapsto f_i(x)$ if $x \in C_i$

$\Rightarrow (C, f)$ is an upper bound.

$\Rightarrow$ Zorn lemma, $\exists (C,f)$ maximal for this order.

- Show $C = B$:

Assume $x \in C \setminus B$. Consider $I = \{r \in R,\ rx \in C\}$ ideal of $R$

Consider $h: I \to Q$ $\quad r \mapsto f(rx)$

$\exists h': R \to Q$ s.t. $h'|_I = h$.

Let $C' := C + \langle x \rangle$, $f': C' \to Q$, $y + rx \mapsto f(y) + h'(r)$

• Well-define: $y_1 + r_1 x = y_2 + r_2 x \Rightarrow \underbrace{y_1 - y_2}_{\in C} = (r_2 - r_1)x$

$\Rightarrow r_1 - r_2 \in I \Rightarrow h'(r_2 - r_1) = h(r_2 - r_1)$

$= f((r_2 - r_1)x) = f(y_1) - f(y_2)$.

$\Rightarrow h'(r_2) + f(y_2) = h'(r_1) + f(y_1)$

• $f'|_C = f$ $\Rightarrow$ contradiction. Thus $C = B$, as desired.

**Fact**: $\prod_{i \in I} Q_i$ injective $\Leftrightarrow$ $\forall i \in I$, $Q_i$ injective (diagram chasing)

Proof:

$\Rightarrow$

$$\begin{tikzcd} I \arrow[d, phantom, "\cap"] \arrow[r, "f"] & Q_i \arrow[r, hook] & \prod Q_i \arrow[l, bend right, "\pi_i"'] \\ R \arrow[ur, dashed] & & \end{tikzcd}$$

$\Leftarrow$

$$\begin{tikzcd} & Q_i \\ A \arrow[ur] \arrow[r] \arrow[dr] & \prod_{i \in I} Q_i \arrow[u] \arrow[d] \\ B \arrow[u, phantom, "\cap"] \arrow[uur, dashed] \arrow[ur, dashed] \arrow[r, dashed] & Q_j \end{tikzcd}$$

**Prop**: We have equivalence

(i) $Q$ injective (ii) $\forall$ $0 \to Q \xrightarrow{f} A \to B \to 0$ exact splits

(iii) $Q$ direct summand of injective module

Proof: (i)$\Rightarrow$(ii) $Q$ injective

$$\begin{tikzcd} Q \arrow[r, "1_Q"] \arrow[d, hook, "f"'] & Q \\ A \arrow[ur, dashed, "\exists g"'] & \end{tikzcd}$$

$g$ makes seq split.

(ii)$\to$(iii): Let $Q \hookrightarrow A$, $A$ injective ($R$-Mod has enough injectives)

$0 \to Q \to A \twoheadrightarrow A/Q \to 0$ exact hence split from (ii)

Thus, $A \cong Q \oplus A/Q$. $\square$

✳ **Example**: $R=\mathbb{Z}$, $k[X]$ or any PID. (vành chính).

- **Prop**: In $\mathbb{Z}$-mod (i.e. abelian group)
  Abelian group $D$ is injective $\Leftrightarrow$ $D$ is divisible

, $D$ is divisible $\Leftrightarrow$ if $y\in D$, $\exists n\in\mathbb{Z}\setminus\{0\}$, $x\in D$ so $nx=y$.

(eg. . $\mathbb{Q}$ is divisible
. quotients of divisible abelian groups are divisible
$\mathbb{Q}/\mathbb{Z}$ are divisible (rational circle). )

- **Proof**: $D$ injective, $y\in D$. Use Baer's criterion:

fix $n$
$$\begin{tikzcd} n\mathbb{Z} \arrow[r, "n\mapsto y"] \arrow[d, phantom, "\cap" rotate=90] & D \\ \mathbb{Z} \arrow[ur, dashed, "1\mapsto x"'] & \end{tikzcd}$$
$\Rightarrow nx=y \Rightarrow D$ divisible

Similarly, if $D$ divisible
$$\begin{tikzcd} n\mathbb{Z} \arrow[r, "n\mapsto y"] \arrow[d, phantom, "\cap" rotate=90] & D \\ \mathbb{Z} \arrow[ur, dashed, "1\mapsto x"'] & \end{tikzcd}$$
(since only ideals of $\mathbb{Z}$ are $n\mathbb{Z}$).

$\Rightarrow D$ injective according to Baer's criterion $\square$

✳ $R=\mathbb{Z}$, every abelian group $A$ can be embedded into divisible abelian group. (i.e. injective $\mathbb{Z}$-module)

**Proof**: Given $A$ abelian group $\quad A^\vee := \mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Q}/\mathbb{Z})$

Pontryagin dual of $A$

$f: A\to\mathbb{Q}/\mathbb{Z} \rightsquigarrow f$ character of $A$
↳ đặc trưng

$\alpha: A\to A^{\vee\vee}$
$a\mapsto \underbrace{(\varphi\mapsto\varphi(a))}_{\alpha(a)}$

- Show $\alpha$ injective: If $a\in A$ $\neq 0$.
  Show $\alpha(a)\neq 0$ $\alpha(a): A^\vee\to\mathbb{Q}/\mathbb{Z}$, i.e. $\exists\varphi: A\to\mathbb{Q}/\mathbb{Z}$ s.t. $\varphi(a)\neq 0$.

* $a$ is not torsion ($na\neq 0$ $\forall n\in\mathbb{N}^*$) Take any $y\neq 0$, $y\in\mathbb{Q}/\mathbb{Z}$

$$\begin{tikzcd} \mathbb{Z}\simeq\langle a\rangle \arrow[r, "a\mapsto y"] \arrow[d] & \mathbb{Q}/\mathbb{Z} \\ A \arrow[ur, dashed, "\exists\varphi"'] & \end{tikzcd}$$

Since $\mathbb{Q}/\mathbb{Z}$ divisible/injective

$\Rightarrow \exists\varphi: A\to\mathbb{Q}/\mathbb{Z}$ s.t. $\varphi(a)=y\neq 0$

* $na = 0$, $n = \mathrm{ord}(a) \in \mathbb{N}^*$, $\langle a\rangle \simeq \mathbb{Z}/n\mathbb{Z}$

$$\begin{tikzcd} \mathbb{Z}/n\mathbb{Z} \simeq \langle a\rangle \arrow[r, "a \mapsto {[\frac{1}{n}]}"] \arrow[d] & \mathbb{Q}/\mathbb{Z} \\ A \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

$\exists \varphi : A \to \mathbb{Q}/\mathbb{Z}$ since $\mathbb{Q}/\mathbb{Z}$ injective $\mathbb{Z}$-mod.

$\varphi(a) = [\frac{1}{n}] \neq 0$.

- Given $A$ abelian group, $\exists$ free ab group $\mathbb{Z}^{(I)} \overset{q}{\twoheadrightarrow} A^\vee \to 0$

$\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z}) \rightsquigarrow 0 \to A^{\vee\vee} \to (\mathbb{Z}^{(I)})^\vee$ injective

But $(\mathbb{Z}^{(I)})^\vee = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{(I)}, \mathbb{Q}/\mathbb{Z}) \cong \prod_I \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$

$\cong (\mathbb{Q}/\mathbb{Z})^I$ divisible as $\mathbb{Q}/\mathbb{Z}$ divisible

- $A \hookrightarrow A^{\vee\vee} \hookrightarrow (\mathbb{Q}/\mathbb{Z})^I$ injective, as desired $\square$

(*) $R$mod can be embedded into injective module:

$R$ - com ring with 1

$I := \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ is $R$-module

with multiplication $(r\varphi)(x) = \varphi(rx)$. — need $R$ to be commutative

- lemma: $I \neq 0$ $\quad \varphi : \mathbb{Z} \to R \quad n \mapsto n \cdot 1$

Case 1: $\varphi$ injective (char $R = 0$) $\quad$ $\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d, hook] & \mathbb{Q}/\mathbb{Z} \\ R \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$ $\Big\}$ prove similar to case of $\mathbb{Z}$-mod

- char $R = n$ $\quad$ $\begin{tikzcd} \mathbb{Z}/n\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Q}/\mathbb{Z} \\ R \arrow[ur, dashed] & \end{tikzcd}$

⊕ $I$ is injective $R$-module:

We have natural isomorphism ($A$ is $R$-module)

$$\mathrm{Hom}_R(A, I) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$$

$$f \longmapsto (a \mapsto f(a)(1_R))$$

$$(a \mapsto (r \mapsto \varphi(ra))) \longleftarrow\!\!\!| \quad \varphi$$

This map has "natural" property:

Given $0 \to A \to B \to C \to 0$ in $R$-mod

exact
$$\begin{array}{ccccccccc} 0 \to & \mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z}) & \to & \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}) & \to & \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) & \to 0 \\ & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & \\ 0 \to & \mathrm{Hom}_R(C, I) & \to & \mathrm{Hom}_R(B, I) & \to & \mathrm{Hom}_R(A, I) & \to 0 \end{array}$$

$\Rightarrow \mathrm{Hom}_R(-, I)$ exact $\Rightarrow I$ injective

$\Leftarrow$ Fix $A \in R$-mod, $Q = \prod_{f \in \mathrm{Hom}_R(A, I)} I$ injective since $I$ injective

$e : A \to Q$

$a \mapsto (f(a))_{f: A \to I}$

Show $e$ injective : $a \in A, a \neq 0$,

$\mathrm{Hom}_R(A, I) \cong \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$

$f \longleftarrow \exists \varphi : A \to \mathbb{Q}/\mathbb{Z}, \varphi(a) \neq 0$ (proven before)

$\forall r \in R \quad f(a)(r) = \varphi(ra) \Rightarrow f(a) \neq 0$ as $f(a) = \varphi(a)$.

$\Rightarrow e(a) \neq 0 \Rightarrow e$ injective

$\Rightarrow A$ is injective $R$-mod. $\square$

# ⊛ Flat module

$A\otimes_R - : R\text{-mod} \to Ab$ right-exact. Then we say $A$ is $\boxed{\text{flat}}$

if $A\otimes_R -$ is exact

$\Leftrightarrow$ if $0 \to M \to N$ exact then $0 \to A\otimes_R M \to A\otimes_R N$ exact

- Lemma: 1) Projective modules are flat.

2) $\bigoplus_{i\in I} A_i$ flat $\Leftrightarrow$ $\forall i$ $A_i$ flat.

Proof: 2) Given $f: B \to C$ injective

$$\begin{tikzcd}
(\bigoplus_i A_i)\otimes_R B \arrow[r, "id\otimes f"] \arrow[d, "\cong"] & (\bigoplus_i A_i)\otimes_R C \arrow[d, "\cong"] \\
\bigoplus_i (A_i\otimes_R B) \arrow[r, "\bigoplus_i (id\otimes f)"] & \bigoplus_I (A_i\otimes_R C)
\end{tikzcd}$$

$id\otimes f : (\bigoplus A_i)\otimes_R B \to (\bigoplus A_i)\otimes_R C$ injective

$\Leftrightarrow$ $id\otimes f : A_i\otimes_R B \to A_i\otimes_R C$ injective $\forall i$

$\Leftrightarrow$ $id\otimes_R f : (\bigoplus A_i)\otimes_R B_i \to (\bigoplus A_i)\otimes C_i$

Thus, (2) is proven

1) - $R$ is flat as:

$$\begin{array}{ccccccccc}
0 & \to & A & \to & B & \to & C & \to & 0 \\
 & & \cong & & \cong & & \cong & & \\
0 & \to & R\otimes_R A & \to & R\otimes_R B & \to & R\otimes_R C & \to & 0
\end{array}$$

$\Rightarrow R^{(I)}$ flat $\forall I$

$\Rightarrow$ Projective modules are flat. (Since it is direct summand of free module) □

## ⊛ Right-derived functors of $\mathrm{Hom}_R(A,-)$, $\mathrm{Ext}^n_R(A,-)$:

exists: $\mathrm{Hom}_R(A,-)$ left exact, $R$-mod has enough injectives

i.e. Given $B$, there is injective resolution

$$0\to B\to I^0\to I^1\to \cdots \quad \text{exact with } I^n \text{ injective}$$

$$\Rightarrow\ 0\to \mathrm{Hom}_R(A,I^0)\to \mathrm{Hom}_R(A,I^1)\to \cdots \quad \text{is complex}$$

$$\mathrm{Ext}^n_R(A,B) = \frac{\ker(\mathrm{Hom}_R(A,I^n))}{\mathrm{Im}(\mathrm{Hom}_R(A,I^{n-1}))}$$

- $\mathrm{Ext}^0_R(A,B) = \mathrm{Hom}_R(A,B)$
  - If $B$ injective $\Leftrightarrow$ $\mathrm{Ext}^n_R(A,B)=0$ $\forall A,\ n\geq 0$
  - If $A$ projective $\Rightarrow$ $\mathrm{Ext}^n_R(A,B)=0$ $\forall n$

  (holds true for general theory for derived functors)

  since $\mathrm{Hom}_R(A,-)$ exact. then with

  $$0\to B\to I^0\to I^1\to \cdots \quad \text{exact then}$$

  $$0\to \mathrm{Hom}_R(A,I^0)\to \mathrm{Hom}(A,I^1)\to\cdots \quad \text{also exact}$$

- Long exact seq: $0\to M\to N\to P\to 0$ exact

$$\rightsquigarrow\ 0\to \mathrm{Hom}_R(A,M)\to \mathrm{Hom}_R(A,N)\to \mathrm{Hom}_R(A,P)\xrightarrow{\delta}$$

$$\mathrm{Ext}^1_R(A,M)\to \mathrm{Ext}^1_R(A,N)\to \mathrm{Ext}^1_R(A,P)\xrightarrow{\delta} \cdots$$

- $\operatorname{Ext}^n_R\left(\bigoplus_i A_i, B\right) \simeq \prod_i \operatorname{Ext}^n_R(A_i, B)$

since $0 \to B \to I^0 \to I^1 \to \cdots$ injective resolution

$\rightsquigarrow 0 \to \operatorname{Hom}_R(A_i, I^0) \to \operatorname{Hom}_R(A_i, I^1) \to \cdots$

$\rightsquigarrow 0 \to \prod_i \operatorname{Hom}_R(A_i, I^0) \to \prod_i \operatorname{Hom}_R(A_i, I^1) \to \cdots$

$\cong$

$\rightsquigarrow 0 \to \operatorname{Hom}_R\left(\bigoplus_i A_i, I^0\right) \to \operatorname{Hom}_R\left(\bigoplus_i A_i, I^1\right) \to \cdots$

- $\operatorname{Ext}^n_R\left(A, \prod B_i\right) \simeq \prod_i \operatorname{Ext}^n_R(A, B_i)$

since $0 \to B_i \to I_i^0 \to I_i^1 \to \cdots$

$\rightsquigarrow 0 \to \prod B_i \to \prod_i I_i^0 \to \prod_i I_i^1 \to \cdots$

Follows from $\operatorname{Hom}_R\left(A, \prod B_i\right) \cong \prod_i \operatorname{Hom}_R(A, B_i)$

⊛ Right-derived functors of $\operatorname{Hom}_R(-, A)$

$\operatorname{Hom}_R(-, A) : R\text{-mod}^{op} \to Ab$ left-exact.

projective in $R$-mod $\Leftrightarrow$ injective in $(R\text{-mod})^{op}$

right derived $R^n \operatorname{Hom}_R(-, A) \simeq \operatorname{Ext}^n_R(-, A)$

double complexes to prove

## ⊛ Left-derived functors of $A\otimes_R -$:

$A\otimes_R - : R\text{-mod}\to Ab$ right exact and $R$-mod has enough proj

$\Rightarrow$ Left-derived functor of $A\otimes_R -$ exists

$$Tor_n^R(A,-) := L_n(A\otimes_R -).$$

i.e. given $0\leftarrow B\leftarrow P_0\leftarrow P_1\leftarrow\cdots$ projective resolution

$$\rightsquigarrow \quad 0\leftarrow A\otimes_R P_0\leftarrow A\otimes_R P_1\leftarrow\cdots$$

$$Tor_n^R(A,B) := \frac{\ker(A\otimes_R P_n\to A\otimes_R P_{n-1})}{Im(A\otimes_R P_{n+1}\to A\otimes_R P_n)}.$$

- $Tor_0^R(A,B)\simeq A\otimes_R B$
- $Tor_n^R(A,B)=0$ if $A$ is flat (i.e. $A\otimes_R -$ exact) $\forall n>0$ or $B$ projective
- with $0\to M\to N\to P\to 0$ exact

$$\cdots\to Tor_1^R(A,M)\to Tor_1^R(A,N)\to Tor_1^R(A,P)\to A\otimes_R M\to A\otimes_R N\to A\otimes_R P\to 0$$

- $Tor_n^R(\bigoplus A_i, B)\simeq\bigoplus_i Tor_n^R(A_i,B)$ $\to$ $\otimes$ commutes with $\oplus$

  $Tor_n^R(A,\bigoplus_i B_i)\simeq\bigoplus_i Tor_n^R(A,B_i)$ $\to$ $\oplus$ projective is also projective.
- $A\otimes_R B\simeq B\otimes_R A \Rightarrow Tor_n^R(A,B)\simeq Tor_n^R(B,A)$

  $\hookrightarrow$ next lecture. double complex

- $\mathrm{Ext}^n_{\mathbb{Z}}(\cdot,\cdot) = 0 \quad \forall n \geq 2$

  $\mathrm{Tor}^n_{\mathbb{Z}}(\cdot,\cdot) = 0 \quad \forall n \geq 2.$

as $\exists$ $0 \hookrightarrow D \hookrightarrow D/A \longrightarrow 0 \longrightarrow 0 \cdots$ is injective resolu

$\phantom{as \exists 0 \hookrightarrow}$ divisible $\quad$ divisible

$0 \longleftarrow A \twoheadleftarrow^{f} L_0 \longleftarrow \ker f \longleftarrow 0 \cdots$ is projec resolution

free $\quad$ free ↳ nontrivial (need Zorn lemma).

## Lecture 28/06/2020 Double complexes, Group cohomology

**Def** (double complex) A double complex $C^{\cdot,\cdot}$ is a diagram in R-mod of the form

$$\begin{tikzcd}
\cdots \arrow[r] & C^{p-1,q-1} \arrow[r] \arrow[d] & C^{p-1,q} \arrow[r, "d_h"] \arrow[d] & C^{p-1,q+1} \arrow[r] \arrow[d, "d_v"] & \cdots \\
\arrow[r] & C^{p,q-1} \arrow[r] \arrow[d] & C^{p,q} \arrow[r] \arrow[d] & C^{p,q+1} \arrow[r] \arrow[d, "d_v"] & \cdots \\
\arrow[r, "d_h"] & C^{p+1,q-1} \arrow[r, "d_h"] \arrow[d] & C^{p+1,q} \arrow[r, "d_h"] \arrow[d] & C^{p+1,q+1} \arrow[r, "d_h"] \arrow[d] & \cdots \\
& {} & {} & {} &
\end{tikzcd}$$

each square is 'anti' commutative $\Rightarrow d_h d_v + d_v d_h = 0$

each row, column is a complex $\begin{cases} d_v^2 = 0 \\ d_h^2 = 0 \end{cases}$

- We say $C^{\cdot,\cdot}$ is bounded if $\forall n$, the diagonal $p+q=n$ have finitely many nonzero terms

e.g. $C^{\cdot,\cdot}$ is positive when $C^{p,q}=0$ when $p<0$ or $q<0$.

- Let $C^{\cdot,\cdot}$ be a bounded double complex, define the total complex (phức toàn phần) $\mathrm{Tot}(C)$ by

when nó positive

$$\forall n \geq 0 \quad \mathrm{Tot}(C)^n := \bigoplus_{p+q=n} C^{p,q} = C^{n,0} \oplus C^{n-1,1} \oplus \dots \oplus C^{0,n}$$

$D: \mathrm{Tot}(C)^n \to \mathrm{Tot}(C)^{n+1}$ by :

$$\begin{tikzcd}
C^{p,q} \arrow[r, "d_h"] \arrow[d, "d_v"] & C^{p,q+1} \\
C^{p+1,q} &
\end{tikzcd}$$

$$d: C^{p,q} \to C^{p+1,q} \oplus C^{p,q+1} \hookrightarrow \mathrm{Tot}(C)^{n+1}$$
$$x \mapsto (d_v x, d_h x) \quad (p+q=n)$$

$$\Rightarrow D: \mathrm{Tot}(C)^n \to \mathrm{Tot}(C)^{n+1}$$
$$C^{n,0} \oplus \dots \oplus C^{0,n} \to C^{n+1,0} \oplus \dots \oplus C^{0,n+1}$$
$$(x_0, \dots, x_n) \mapsto (d_v x_0, d_h x_0 + d_v x_1, \dots, d_h x_{n-1} + d_v x_n, d_h x_n)$$

- **Fact**: $(\mathrm{Tot}(C)^{\cdot}, D)$ is a complex.

$D^2 = 0$. $\quad D^2(x_0, \dots, x_n) = D(d_v x_0, d_h x_0 + d_v x_1, \dots, d_h x_n)$
$$= (d_v d_v x_0, d_h d_v x_0 + d_v d_h x_0 + d_v d_v x_1, \dots, d_h d_h x_n)$$

- Given $C^{\bullet,\bullet}$ positive double complex. We augment it by add $H_v^0$, $H_h^0$ as follows:

$$\begin{tikzcd}
 & H_v^0(C^{0,0}) \arrow[r, "d_h"] \arrow[d] & H_v^0(C^{0,1}) \arrow[r, "d_h"] \arrow[d] & \cdots \\
H_h^0(C^{0,0}) \arrow[r, "d_h"] \arrow[d, "d_v"'] & C^{0,0} \arrow[r, "d_h"] \arrow[d, "d_v"] & C^{0,1} \arrow[r] \arrow[d, "d_v"] & \cdots \\
H_h^0(C^{1,0}) \arrow[r] \arrow[d, "d_v"'] & C^{1,0} \arrow[r, "d_h"] \arrow[d] & C^{1,1} \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

$$H_v^0(C^{0,q}) := \ker(d_v : C^{0,q} \to C^{1,q})$$
$$H_h^0(C^{q,0}) := \ker(d_h : C^{q,0} \to C^{q,1})$$

If all rows and columns of $C^{\bullet,\bullet}$ are exact then we have cannonical isomorphisms

$$\begin{tikzcd}
 & H_v^0(C^{0,0}) \arrow[r, "d_h"] \arrow[d, "d_v"] & H_v^0(C^{0,1}) \arrow[r] \arrow[d] & {} \\
H_h^0(C^{0,0}) \arrow[r, hook, "d_h"] \arrow[d, "d_v"] & C^{0,0} \arrow[r, "d_h"] \arrow[d, "d_v"] & C^{0,1} \arrow[r] \arrow[d] & {} \\
H_h^0(C^{1,0}) \arrow[r, hook] \arrow[d, "d_v"] & C^{1,0} \arrow[r] \arrow[d] & C^{1,1} \arrow[r] \arrow[d] & {} \\
{} & {} & {} &
\end{tikzcd}$$

cohomology of columns

$$H_v^n(H_h^0(C^{\bullet,0})) \simeq H_h^n(H_v^0(C^{0,\bullet})) \simeq H^n(\mathrm{Tot}(C)^\bullet)$$

(double complex lemma)

- Proof: Define $f : H_h^0(C^{n,0}) \to \mathrm{Tot}(C)^n$ by $\hookrightarrow C^{n,0}$, $D$

then:

if $x \in \ker(d_v : H_h^0(C^{n,0}) \to H_h^0(C^{n+1,0})) \mapsto (x, 0, \dots, 0) \in C^{n,0}$

$$\Rightarrow D(x, 0, \dots, 0) = (d_v x, d_h x + d_v 0, \dots, d_h 0) = 0$$

If $x \in \mathrm{Im}(d_v : H_h^0(C^{n-1,0}) \to H_h^0(C^{n,0})) \Rightarrow x = d_v y$, $y \in C^{n-1,0}$

and $d_h y = 0$

$\Rightarrow \quad (x, 0, \ldots, 0) = (d_v y, d_h y + d_v 0, \ldots, d_h 0) = D(y, 0, \ldots, 0)$

$\Rightarrow$ $f$ turns $n$-cycle of $H_h^0(C^{\bullet,0})$ into $n$-cycle of $\mathrm{Tot}(C)^{\bullet}$

$n$-boundary of $H_h^0(C^{\bullet,0})$ into $n$-boundary of $\mathrm{Tot}(C)^{\bullet}$

$\Rightarrow$ induces $\phi: H_v^n(H_h^0(C^{\bullet,0})) \to H^n(\mathrm{Tot}(C)^{\bullet})$

$[x] \mapsto [(x, 0, \ldots, 0)]$ homomorphism

- Prove $\phi$ is bijective, i.e. $\forall$ class $[(x_0, \ldots, x_n)] \in H^n(\mathrm{Tot}(C)^{\bullet})$,

$\exists!$ class $[x] \in H_v^n(H_h^0(C^{\bullet,0}))$ s.t. $[(x, 0, \ldots, 0)] = [(x_0, \ldots, x_n)]$.

Existence: $\forall 1 \le k \le n$, $\forall$ class $[(x_0, \ldots, x_k, 0, \ldots, 0)]$

show $\exists$ suitable $y_0, \ldots, y_{k-1}$ s.t. $[(x_0, \ldots, x_k, 0, \ldots, 0)] = [(y_0, \ldots, y_{k-1}, 0, \ldots, 0)]$

Let $\pi: \mathrm{Tot}(C)^{n+1} \to C^{n-k, k+1}$ be canonical projection

given $[(x_0, \ldots, x_k, 0, \ldots, 0)] \in H^n(\mathrm{Tot}(C)^{\bullet})$.

as $x_k \in C^{n-k, k+1} \Rightarrow d_h(x_k) = d_h(x_k) + d_v(0) = \pi(D(x_0, \ldots, x_k, 0, \ldots, 0)) = 0$

also, the $(n-k)$th row of $C^{\bullet,\bullet}$ is exact $\Rightarrow x_k = d_h(z)$

for some $z \in C^{n-k, k-1}$.

$\Rightarrow (x_0, \ldots, x_k, 0, \ldots, 0) - (x_0, \ldots, x_{k-1} - d_v z, 0, \ldots, 0)$

$= (0, \ldots, 0, d_v z, d_h z, 0, \ldots, 0) = D(0, \ldots, z, 0, \ldots, 0)$

$\Rightarrow [x_0, \ldots, x_k, 0, \ldots, 0] = [(x_0, \ldots, x_{k-1} - d_v z, 0, \ldots, 0)]$.

By induction on $k$, we are done.

$= \phi([z'])$

**Uniqueness:** If $[(x_0,\dots,x_n)] = [(z,0,\dots,0)] = [(z',0,\dots,0)] \in H^n_v(\mathrm{Tot}\,C)$

with $z, z' \in C^{n,0}$ s.t. $d_v z = d_v z' = 0$.

$\Rightarrow (z,0,\dots,0) - (z',0,\dots,0) = D(y_0,\dots,y_{n-1})$ for some $(y_0,\dots,y_{n-1}) \in \mathrm{Tot}(C)^{n-1}$

$\Rightarrow z - z' = d_v y_0 \Rightarrow [z] = [z']$ in $H^n_v(H_h(C^{\bullet,0}))$

$\Rightarrow$ the class $[z]$ is unique. $\square$

**Examples:** $A, B$ $R$-modules

$A \xleftarrow{\varepsilon} P_\bullet$ projective resolution

$B \xrightarrow{\eta} I_\bullet$ injective ——

$$\begin{tikzcd}
 & \mathrm{Hom}_R(A,I^0) \arrow[r] \arrow[d, "\varepsilon^*"] & \mathrm{Hom}_R(A,I^1) \arrow[r] \arrow[d, "\varepsilon^*"] & \cdots \\
\mathrm{Hom}_R(P_0,B) \arrow[r] \arrow[d] & \mathrm{Hom}_R(P_0,I^0) \arrow[r] \arrow[d] & \mathrm{Hom}_R(P_0,I^1) \arrow[r] \arrow[d] & {} \\
\mathrm{Hom}_R(P_1,B) \arrow[r, "\eta_*"] & \mathrm{Hom}_R(P_1,I^0) \arrow[r] \arrow[d] & \mathrm{Hom}_R(P_1,I^1) \arrow[r] \arrow[d] & {} \\
 & \vdots & \vdots &
\end{tikzcd}$$

Rows $\mathrm{Hom}_R(P_i, I^\bullet)$ is exact since $P_i$ is projective

Columns $\mathrm{Hom}_R(P_\bullet, I^i)$ ——————— $I^i$ is injective.

(show $\eta_*, \varepsilon_*$ injective)

By double complex lemma, $H^n(\mathrm{Hom}_R(A,I^\bullet)) \cong H^n(\mathrm{Hom}_R(P_\bullet, B))$

$\Rightarrow R^n(\mathrm{Hom}_R(-,B))(A) \cong R^n(\mathrm{Hom}_R(A,-))(B)$

$\mathrm{Ext}^n_R(A,B) \simeq \mathrm{Ext}^n_R(A,B)$

$\Rightarrow \mathrm{Ext}^n_R(A,B)$ can be computed by using projective resolutions of $A$ or injective resolution of $B$. $\lrcorner$

Similarly, $A \leftarrow P_\bullet$ and $B \leftarrow Q_\bullet$ projective resolutions.

$$\begin{tikzcd}
 & A\otimes_R Q_0 & A\otimes_R Q_1 \arrow[l] & \cdots \arrow[l] \\
P_0\otimes_R B & P_0\otimes_R Q_0 \arrow[l] \arrow[u] & P_0\otimes_R Q_1 \arrow[l] \arrow[u] & \cdots \arrow[l] \\
P_1\otimes_R B \arrow[u] & P_1\otimes_R Q_0 \arrow[l] \arrow[u] & P_1\otimes_R Q_1 \arrow[l] \arrow[u] & \cdots \arrow[l] \\
 & \arrow[u] & \arrow[u] &
\end{tikzcd}$$

By double complex lemma

$$L_n(B\otimes_R -)(A) \simeq L_n(A\otimes_R -)(B)$$

$$\Rightarrow \quad \mathrm{Tor}^R_n(B,A) \simeq \mathrm{Tor}^R_n(A,B) \quad \text{balancing Tor.}$$

$\circledast$

Let $(C_\bullet, d)$ and $(C'_\bullet, d')$ be positive complexes form the double complex

$$\begin{tikzcd}
C_{p+1}\otimes_R C'_{q+1} \arrow[r, "(-1)^{p+1}\mathrm{id}\otimes d"] \arrow[d, "d\otimes \mathrm{id}"] & C_{p+1}\otimes C'_q \arrow[r] \arrow[d, "d\otimes \mathrm{id}"] & \cdots \\
C_p\otimes_R C'_{q+1} \arrow[r] \arrow[d] & C_p\otimes C'_q \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots &
\end{tikzcd}$$

Define $C\otimes_R C'$ = total complex of this double complex i.e. $(C \otimes_R C')_n = \bigoplus_{i=0}^{n} (C_{n-i} \otimes C'_i)$

where $D: (C \otimes_R C')_{n+1} \to (C \otimes_R C')_n$

$$c \otimes c' \mapsto dc \otimes c' + (-1)^p c \otimes dc'$$

with $c \in C_p$, $c' \in C'_q$, $p+q = n+1$.

- If $c \in Z^p(C)$, $c' \in Z^q(C') \Rightarrow dc = dc' = 0$

$\Rightarrow D(c \otimes c') = dc \otimes c' + (-1)^p c \otimes dc' = 0$.

$\Rightarrow c \otimes c' \in Z^{p+q}(C \otimes_R C')$

- If $c \in Z^p(C)$, $c' \in B^q(C')$ or $c \in B^p(C)$, $c' \in Z^q(C')$ $\Rightarrow c \otimes c' \in B^{p+q}(C \otimes_R C')$.

$\Rightarrow H_p(C) \otimes_R H_q(C') \longrightarrow H_{p+q}(C \otimes_R C')$ Homology product

$$\frac{Z^p(C)}{B^p(C)} \otimes \frac{Z^q(C')}{B^q(C')} \simeq \frac{Z^p(C) \otimes Z^q(C')}{Z^p(C) \otimes B^q(C') + B^p(C) \otimes Z^q(C')}$$

When $p, q$ vary s.t. $p+q = n$, we have a map

$$\bigoplus_{p+q=n} H_p(C) \otimes_R H_q(C') \to H_n(C \otimes_R C').$$

$$[c] \otimes [c'] \mapsto [c \otimes c']$$

These two are not "equal"

- **Künneth formula.** We have natural exact seq. (if $R$ is PID, $C_i$ free)

$$0 \to \bigoplus_{p+q} H_p(C) \otimes_R H_q(C') \to H_n(C \otimes C') \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(C), H_q(C')) \to 0$$

• Proof (sketch).

Case 1: $d$ of $C = 0$. then $H_p(C_n) = C_n$ is free $\forall n \geq 0$

$\Rightarrow$ projective $\Rightarrow$ flat $\Rightarrow \mathrm{Tor}_1(H_p(C), H_q(C')) = 0 \ \forall p, q$

then $D: (C \otimes_R C')_{n+1} \to (C \otimes_R C')_n$

$$c \otimes c' \longmapsto (-1)^p c \otimes d c' \quad \forall c \in C_p, \ c' \in C'_q$$

can obtain

$$C = \bigoplus_{i \in I} R \text{ free} \rightsquigarrow H_n(C \otimes C') \simeq \bigoplus_{p+q=n} H_p(C) \otimes_R H_q(C').$$

Case 2: General case. $Z_p = Z_p(C)$, $B_p = B_p(C)$

If $C_p$ free $\Rightarrow Z_p, B_p$ free ($R$ is PID).

Let $Z$ be complex $Z_0 \xleftarrow{0} Z_1 \xleftarrow{0} Z_2 \leftarrow \cdots$

$B$ ——— $B_0 \xleftarrow{0} B_1 \xleftarrow{0} B_2 \leftarrow$

then $\forall p$ $\quad 0 \to Z_p \to C_p \xrightarrow{d} \underbrace{B_{p-1}}_{\text{free} \Rightarrow \text{flat}} \to 0$

$$\Rightarrow \ 0 \overset{\mathrm{Tor}_1(B_{p-1}, C'_q)}{=} 0 \to Z_p \otimes C'_q \to C_p \otimes C'_q \to B_{p-1} \otimes C'_q \to 0$$

exact

$$\Rightarrow \bigoplus_{p+q=n} B_p \otimes H_q(C') \xrightarrow{\delta_n} \bigoplus_{p+q=n} Z_p \otimes H_q(C')$$

$$H_n(C \otimes C') \longrightarrow \bigoplus_{p+q=n-1} B_{p-1} \otimes H_q(C') \xrightarrow{\delta_{n-1}} \cdots$$

$$\delta(b \otimes [c']) = b \otimes [c']$$

$$0 \to \operatorname{coker} \delta_n \to H_n(C \otimes C') \to \ker \delta_{n-1} \to 0 \ \text{exact}$$

$\qquad\qquad \|\qquad\qquad\qquad\qquad\qquad\qquad \|$

$$\bigoplus_{p+q} H_p(C)\otimes H_q(C') \qquad \bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(C), H_q^{?}(C'))$$

$Z_p$ free $\Rightarrow \mathrm{Tor}_1(Z_p, H_q(C')) = 0$

from $0 \to B_p \to Z_p \to H_p(C) \to 0$

we have

$$0 \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(C), H_q(C'))$$

$$\bigoplus_{p+q=n-1} B_p\otimes H_q(C') \to \bigoplus_{p+q=n-1} Z_p\otimes H_q(C') \to \bigoplus_{p+q=n-1} H_p(C)\otimes H_q(C') \to 0$$

$\lrcorner$

⊛ homology with coefficients $\quad C_\bullet$ complex, $M$ $R$-module

$$C_0 \leftarrow C_1 \leftarrow \cdots \Rightarrow C_0\otimes_R M \leftarrow C_1\otimes_R M \leftarrow \cdots$$

$$H_n(C_\bullet; M) := H_n(C_\bullet\otimes_R M) \text{ homology of } C \text{ with coef in } M.$$

- Künneth for $C_0' = M$, $C_n' = 0$ $\forall n\neq 0$

  $C_i$ free, $R$ is PID

  $$\Rightarrow 0 \to H_n(C)\otimes_R M \to H_n(C; M) \to \mathrm{Tor}_1(H_{n-1}(C), M) \to 0$$

  universal coefficient theorem

  (Calculate homology with coef in $M$ by computing

  $H_n(C)\otimes_R M$ and $\mathrm{Tor}_1 \cdots$)

- When $R$ field $\mathrm{Tor}_1 = 0 \Rightarrow H_n(C)\otimes_R M \cong H_n(C; M)$.

# ⊛ Group Cohomology: $G$ group.

**Def.** A G-module is abelian group $M$ with action of $G$,
i.e. map $G\times M\to M$ $(g,x)\mapsto g.x$

s.t. $\begin{cases} 1_G x = x & \forall x\in M \\ g(h.x) = (gh)x & \forall g,h\in G, \forall x\in M \\ g(x+y) = g.x+g.y & \forall g\in G, \forall x,y\in M \end{cases}$

G-module $\Leftrightarrow$ $\mathbb{Z}[G]$-module

group ring of $G$ $=\left\{\sum_{g\in G} \lambda_g g : \lambda_g\in\mathbb{Z}\right\}$

- A hom of G-modules $M\to N$ is $f:M\to N$ s.t.
  $f(x+y)=f(x)+f(y)$ and $f(gx) = xf(x)$.

- $A$ is G-module,
  $A^G=\{x\in A : g.x=x \ \forall g\in G\}$ invariant submodule
  $A_G = A/\langle g.x-x : g\in G, a\in A\rangle$ coinvariant module

$f: A\xrightarrow{G\text{-mod}} B \Rightarrow f(A^G)\subseteq B^G$. Define $f^G: A^G\to B^G$ be $f|_{A^G}$.

Similarly, $f_G: A_G\to B_G$ $[x]\mapsto[f(x)]$.

- We have 2 functors: $(-)^G: G\text{-mod}\to Ab$
  $(-)_G: G\text{-mod}\to Ab$

Take $\mathbb{Z}$: G-module ($g.n=n$ $\forall g\in G, n\in\mathbb{Z}$).

$\mathbb{Z}[G]\xrightarrow{\varepsilon}\mathbb{Z}$ augmentation map

$\varepsilon\left(\sum_{g\in G} n_g g\right) = \sum_{g\in G} n_g$. $\ker\varepsilon = I_G = \langle g-1_G : g\in G\rangle$
augmentation ideal

We have
$$\begin{tikzcd} A \arrow[r, "\simeq"] \arrow[d, "f"'] & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, A) \arrow[d, "f_*"] \\ B \arrow[r, "\simeq"] & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, B) \end{tikzcd}$$
$x \mapsto (n \mapsto nx)$.
$f(1) \leftarrow\!\shortmid f$

$$\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A) = \{ f \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, A) : \forall g \in G, \forall x \in A \text{ we have } \underbrace{f(g.1)}_{= f(1)} = g f(1) \}$$
$$\simeq \{ x \in A : g.x = x \ \forall g \in A \} = A^G.$$

$\Rightarrow$ we have isomorphism of functors
$$\underline{(-)^G \simeq \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, -)}$$

$$\begin{tikzcd} A^G \arrow[r, "\sim"] \arrow[d, "f^G"'] & \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A) \arrow[d, "f_*"] \\ B^G \arrow[r, "\sim"] & \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, B) \end{tikzcd}$$

- Similarly, $A_G = A / \langle g.x - x : g \in G, x \in A \rangle = A / I_G A$
$$I_G = \ker(\varepsilon : \mathbb{Z}[G] \to \mathbb{Z})$$
$$= \langle g - 1_G : g \in G \rangle$$
$$\simeq \mathbb{Z}[G]/I_G \otimes_{\mathbb{Z}[G]} A$$
$$\simeq \mathbb{Z} \otimes_{\mathbb{Z}[G]} A.$$
$A \otimes_{\mathbb{Z}G} B : g(x \otimes y) = gx \otimes gy$

$\Rightarrow (-)_G \simeq \mathbb{Z} \otimes_{\mathbb{Z}[G]} -$.

- Attention: When $G = \mathbb{Z}$, to avoid confusion between $\mathbb{Z}$-module and $\mathbb{Z}[G]$-module, we denote $G = \mathbb{Z} = C_\infty = \langle \sigma \rangle$.

$$G\text{-module} = \mathbb{Z}[C_\infty]\text{-module} = \mathbb{Z}[\sigma, \sigma^{-1}]\text{-module}$$

if $G = \mathbb{Z}/n\mathbb{Z} = \langle \sigma \mid \sigma^n = 1 \rangle = C_n$

$$\mathbb{Z}[C_n] = \mathbb{Z}[\sigma] / \langle \sigma^n - 1 \rangle.$$

- $(-)^G \simeq \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, -)$ : $G\text{-mod} \to Ab$. is left-exact.

  $A$ $G$-module

  Define $H^n(G, A) := R^n(-)^G A = \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, A)$.

  (cohomology of $G$ with coef in $A$).

  $H_n(G, A) := L_n(-)_G A = \mathrm{Tor}^{\mathbb{Z}[G]}_n(\mathbb{Z}, A)$.

  (homology of $G$ with coef in $A$)

- To compute $H^n(G, A) = \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, A)$:

  1) Choose an $\mathbb{Z}[G]$-injective resolution of $A$

  $\hookrightarrow$ (has enough injective, proof later).

$$0 \to A \to I^0 \to I^1 \to \cdots$$

$$\Rightarrow 0 \to (I^0)^G \to (I^1)^G \to \cdots$$

$$\Rightarrow H^n(G, A) = \frac{\ker((I^n)^G \to (I^{n+1})^G)}{\mathrm{Im}((I^{n-1})^G \to (I^n)^G)}.$$

  (injective resolution is hard to compute)

  2) Choose $\mathbb{Z}[G]$-projective resolution of $\mathbb{Z}$

  canonical free resolution $\nearrow$ can choose to be free

  does not depend on $A$

$$0 \leftarrow \mathbb{Z} \leftarrow P_0 \leftarrow P_1 \leftarrow \dots$$

$$0 \to \mathrm{Hom}_{\mathbb{Z}[G]}(P_0, A) \to \mathrm{Hom}_{\mathbb{Z}[G]}(P_1, A) \to \dots$$

$$\Rightarrow H^n(G, A) = \frac{\ker\left(\mathrm{Hom}_{\mathbb{Z}[G]}(P_n, A) \to \mathrm{Hom}_{\mathbb{Z}[G]}(P_{n+1}, A)\right)}{\mathrm{Im}(\dots)}.$$

Lecture (06/07/2020) Group cohomology (continued)

Induced functor / modules

Shapiro's lemma

Homology group of G with coefficient in G-mod

- G group

⊛ A, B G-module then $Hom_{\mathbb{Z}}(A, B)$ is G-module

$\varphi : A \to B$ (hom of abelian groups)

$g \in G \quad (g\varphi)(x) = g\,\varphi(g^{-1}x) \ \forall x \in A.$

⊛ Induced modules: $H \leq G$ — mô đun cảm sinh

H-modules ⇝ G-module ?

$$\mathrm{Ind}_H^G A := \{\varphi : G \to A \text{ s.t. } \varphi(hg) = h\varphi(g) \ \forall h \in H, \forall g \in G\}$$

is abelian group: $(\varphi + \psi)(g) := \varphi(g) + \psi(g)$.

$\mathrm{Ind}_H^G A$ is G-module: Let $\varphi \in \mathrm{Ind}_H^G A$ $(\varphi : G \to A)$

$g \in G$: $g \cdot \varphi : G \to A$

$(g\cdot\varphi)(x) = \varphi(xg)$.

- check $g\cdot\varphi \in \mathrm{Ind}_H^G A$: $\forall h \in H$

$$(g\cdot\varphi)(hx) = \varphi(hxg) = h\varphi(xg) = h[(g\varphi)(x)].$$

$\Rightarrow \mathrm{Ind}_H^G A$ is a G-module.

- $A \xrightarrow{f} \mathrm{Ind}_H^G A$ if A is G-module

$x \longmapsto f(x) : G \to A \qquad f(x)(g) = g\cdot x.$

It is injective: If $f(x) = 0 \Rightarrow f(x)(g) = g\cdot x = 0 \ \forall g \Rightarrow x = f(x)(1_G) = 0$

It is $G$-module homomorphism

$f(A) = \mathrm{Ind}_G^G A \subseteq \mathrm{Ind}_H^G A$.

- Let $\alpha: A \to B$ be homomorphism of $H$-modules

$\rightsquigarrow$ $\alpha_*: \mathrm{Ind}_H^G A \to \mathrm{Ind}_H^G B$

$\varphi \mapsto \alpha \circ \varphi$

<u>Claim</u>: $\alpha_*$ is hom of $G$-modules.

- $\alpha \circ (\varphi + \psi) = \alpha \circ \varphi + \alpha \circ \psi$
- $(\alpha(g \cdot \varphi))(x) = \alpha((g \cdot \varphi)x) = \alpha(\varphi(xg)) = (\alpha \circ \varphi)(xg)$
$= (g \cdot (\alpha \circ \varphi))(x)$.

$\Rightarrow \alpha \circ (g \cdot \varphi) = g \cdot (\alpha \circ \varphi)$

$\Rightarrow \alpha_*(g \cdot \varphi) = g \cdot (\alpha_* \varphi)$.

$\Rightarrow \mathrm{Ind}_H^G : \mathrm{Mod}_H \to \mathrm{Mod}_G$ $\quad A \rightsquigarrow \mathrm{Ind}_H^G A$

$(A \xrightarrow{\alpha} B) \mapsto \mathrm{Ind}_H^G A \xrightarrow{\alpha_*} \mathrm{Ind}_H^G B$.

- <u>Lemma</u>: We have natural isomorphism of abelian groups

$$\mathrm{Hom}_G(A, \mathrm{Ind}_H^G B) \cong \mathrm{Hom}_H(A, B).$$

where $A$ is $G$-module $\quad H \leq G$

$B$ is $H$ module

<u>Proof</u>: $\mathrm{Hom}_G(A, \mathrm{Ind}_H^G B) \to \mathrm{Hom}_H(A, B)$

- given $\alpha: A \to \mathrm{Ind}_H^G B$

define $\beta: A \to B$ by $\beta(x) = \alpha(x)(1_G)$.

$\Rightarrow \forall h \in H, \forall x \in A$: $\beta(hx) = \alpha(h.x)(1_G)$

$= (h \cdot \alpha(x))(1_G) = \alpha(x)(1_G \cdot h) = \alpha(x)(h \cdot 1_G)$

$= h \cdot \alpha(x)(1_G) = h\,\beta(x).$

- Conversely, given $\beta \in \mathrm{Hom}_H(A, B)$.

  Define $\alpha: A \to \mathrm{Ind}_H^G B$ by $\forall x \in A$

  $\alpha(x)(g) = \beta(g.x)$. is $G$-mod hom

  - These two maps are inverses $\square$

- This isomorphism is natural:

$$\begin{array}{ccc} \mathrm{Hom}_G(A, \mathrm{Ind}_H^G(B)) & \xrightarrow{\sim} & \mathrm{Hom}_H(A, B) \\ \downarrow {\scriptstyle \beta_* \circ - \circ \alpha} & & \downarrow {\scriptstyle \beta \circ - \circ \alpha} \\ \mathrm{Hom}_G(A', \mathrm{Ind}_H^G B') & \xrightarrow{\sim} & \mathrm{Hom}_H(A', B') \end{array}$$

$\forall \alpha: A' \to A$ , $\forall \beta: B \to B'$ $H$map
G-map.

- Universal properties:

$$\mathrm{Hom}_G(A, \mathrm{Ind}_H^G B) \xrightarrow{\sim} \mathrm{Hom}_H(A, B)$$
$$\alpha \longmapsto (x \mapsto \alpha(x)(1_G)).$$

We have a map $\Phi: \mathrm{Ind}_H^G \to B$ ; $\varphi \mapsto \varphi(1_G)$.

$\forall$ G-module $A$

$\forall$ H-module $\beta: A \to B$

$\exists!$ G-mod : $\alpha: A \to \mathrm{Ind}_H^G B$

$\Phi \circ \alpha = \beta$.

$$\begin{tikzcd} A \arrow[d, "\exists!\ \text{G-map}\ \alpha"'] \arrow[dr, "\forall\ \text{H-map}\ \beta"] & \\ \mathrm{Ind}_H^G B \arrow[r, "\Phi"'] & B \end{tikzcd}$$

- Lemma: Functor $\mathrm{Ind}_H^G : \mathrm{Mod}_H \to \mathrm{Mod}_G$ is exact.

Proof: Given $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ exact in $\mathrm{Mod}_H$.

We prove that $0 \to \mathrm{Ind}_H^G(A) \xrightarrow{\alpha_*} \mathrm{Ind}_H^G(B) \xrightarrow{\beta_*} \mathrm{Ind}_H^G(C) \to 0$
exact

i) $\alpha_*$ injective : given $\varphi \in \mathrm{Ind}_H^G(A)$ $\quad \varphi : G \to A$.
s.t. $\alpha \circ \varphi = 0 \Rightarrow \varphi = 0$ (since $\alpha$ injective)

ii) $\beta_* \circ \alpha_* = (\beta \circ \alpha)_* = 0$

iii) $\ker \beta_* \subseteq \mathrm{Im}\, \alpha_*$ : given $\psi \in \mathrm{Ind}_H^G(B)$ s.t.
$\psi \in \ker \beta_* \Leftrightarrow \beta \circ \psi = 0 \Rightarrow \beta(\psi(g)) = 0 \ \forall g \in G$.
$\Rightarrow \psi(g) = \alpha(\varphi(g))$ for some $\varphi : G \to A$.
then $\psi = \alpha_* \varphi$

Show $\varphi \in \mathrm{Ind}_H^G A$ : $h \in H$, $g \in G$
$\Rightarrow \alpha(\varphi(hg)) = \psi(hg) = h\psi(g) = h \cdot \alpha(\varphi(g))$
$= \alpha(h \cdot \varphi(g))$
$\Rightarrow \varphi(hg) = h\varphi(g)$ since $\alpha$ injective.
$\Rightarrow \varphi \in \mathrm{Ind}_H^G A$.

iv) $\beta_*$ surjective. Write $G = \bigsqcup_{s \in S} Hs$. given $\psi \in \mathrm{Ind}_H^G C$.
Since $\beta : B \to C$ surjective $\Rightarrow$ choose $y_s \in B$ s.t. $\beta(y_s) = \psi(s)$.
$\forall s \in S$.
Define $\varphi : G \to B$ by $\varphi(hs) := h \cdot y_s$
$h \in H, s \in S$
$\Rightarrow \varphi \in \mathrm{Ind}_H^G B$ and $\beta_*(\varphi) = \psi$. $\square$

- If $H=\{1\}$, write $\mathrm{Ind}^G A$ for $\mathrm{Ind}_H^G A$.
  and we say $G$-module $A$ is induced if $A \simeq \mathrm{Ind}^G A_0$ for some abelian group $A_0$

1) $A$ is induced $G$-module $\Leftrightarrow$ $\exists A_0 \subseteq A$ abelian group s.t. $A = \bigoplus_{g\in G} g.A_0$.

$\mathrm{Ind}^G A_0 \simeq \mathbb{Z}[G] \otimes_{\mathbb{Z}} A_0$ ($G$-module).

$$g(z\otimes x) := (gz)\otimes x.$$

2) If $G = \coprod_{s\in H} Hs$, $A = \bigoplus_{g\in G} gA_0 = \bigoplus_{h\in H} h.\left(\bigoplus_{s\in H} s.A_0\right)$.

$\Rightarrow$ If $A$ is induced $G$-module then $A$ is also induced $H$-module.

3) $\pi : \mathrm{Ind}^G A \twoheadrightarrow A$ $\quad A$-$G$mod

$$\varphi : G \to A \mapsto \sum_{g\in G} g.\varphi(g^{-1})$$

- Prop: $\mathrm{Mod}_G \simeq \mathrm{Mod}_{\mathbb{Z}[G]}$ has enough injectives
  ($\mathbb{Z}[G]$ not commutative and we only show this for $R$ comm so far).

  $\mathrm{Ind}_H^G$ preserves injectives

Proof: $G=\{1\} \to \mathrm{Mod}_G = \mathrm{Ab}$ which has enough injectives.
- Let $A$ be any $G$-module, $A \hookrightarrow I$ (divisible abelian group)
- Recall $A \hookrightarrow \mathrm{Ind}^G A \hookrightarrow \mathrm{Ind}^G I$ ($\mathrm{Ind}^G$ is exact)
  $x \mapsto (g \mapsto gx)$

It suffices to show that $\mathrm{Ind}^G I$ is injective module

- More generally, if $I$ is an injective $H$-module ; $H \leq G$, then $\mathrm{Ind}_H^G I$ is an injective $G$-module.

Consider $\underbrace{\mathrm{Hom}_G(-, \mathrm{Ind}_H^G I)}_{\text{exact}} \simeq \underbrace{\mathrm{Hom}_H(-, I)}_{\text{exact since } I \text{ injective}}$

$\Rightarrow \mathrm{Ind}_H^G I$ injective $G$-mod. $\square$

⊛ Shapiro's lemma :

$A$: $G$-module $\quad A^G = \{x \in A : g.x = x \ \forall g \in G\}$.

$(-)^G : \mathrm{Mod}_G \to \mathrm{Ab}$ $\qquad$ $\mathbb{Z}$ is $G$-mod

$\cong$ $\qquad$ $G \curvearrowright \mathbb{Z}$ $\quad g.n = n \ \forall g \in G$

$\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, -)$

$\Rightarrow H^n(G, A) := R^n(-)^G(A) = \mathrm{Ext}^n_{\mathbb{Z}[G]}(G, A)$.

Shapiro's lemma : $H^n(G, \mathrm{Ind}_H^G A)) = H^n(H, A)$,

$\forall n \geq 0$, $\forall$ $H$-module $A$.

Proof : $n = 0$ : $H^0(G, \mathrm{Ind}_H^G(A)) \simeq \mathrm{Hom}_G(\underbrace{\mathbb{Z}, \mathrm{Ind}_H^G A}_{(\mathrm{Ind}_H^G A)^G}$

$\simeq \mathrm{Hom}_H(\underbrace{\mathbb{Z}, A}_{A^G}) \simeq H^0(H, A)$.

$n > 0$ Choose injective resolution of $A$ in $\mathrm{Mod}_H$.

$0 \to A \to I^0 \to I^1 \to \cdots$

Since $\mathrm{Ind}_H^G$ is exact and preserves injectives

$0 \to \mathrm{Ind}_H^G A \to \mathrm{Ind}_H^G I^0 \to \mathrm{Ind}_H^G I^1 \to \cdots$

is injective resolution of $\mathrm{Ind}_H^G A$ in $\mathrm{Mod}_G$.

$$\Rightarrow H^n(G, \mathrm{Ind}_H^G A) \overset{\text{def}}{=} H^n\left(\left(\mathrm{Ind}_H^G(I^\cdot)\right)^G\right).$$

$$\overset{n=0}{\simeq} H^n\left((I^\cdot)^H\right) \overset{\text{def}}{=} H^n(H, A). \qquad \square$$

- **Corollary**: If $A = \mathrm{Ind}^G A_0$ induced module then
$$H^n(G, A) = H^n(\{1\}, A_0) = 0 . \quad n \geq 1.$$
since $(-)^{\{1\}} = \mathrm{id} : Ab \to Ab$ is exact
(ie. induced modules are acyclic).

- **Remark.** A short exact seq of $G$-modules $0 \to A \to B \to C \to 0$ induces a long exact seq

$$0 \to A^G \to B^G \to C^G$$
$$\to H^1(G, A) \to H^1(G, B) \to H^1(G, C)$$
$$\to \cdots$$

- If $B$ is acyclic $\Rightarrow$ $0 \to A^G \to B^G \to C^G \to H^1(G, A) \to 0$
$$\cdots\cdots$$
$$H^n(G, B) = 0 \to H^n(G, C) \xrightarrow{\delta} H^{n+1}(G, A) \xrightarrow{n \geq 1} 0$$
$$\Rightarrow H^n(G, C) \simeq H^{n+1}(G, A) \quad n \geq 1$$

- In particular, if $A$ is $G$-module $B = \mathrm{Ind}^G A$ acyclic
$C = B/A$ then exact $0 \to A \to B \to C \to 0$
$$\Rightarrow H^n(G, C) = H^{n+1}(G, A) \quad n \geq 1.$$

- More generally, $\forall\ 0 \to A \to B^1 \to \dots \to B^s \to C \to 0$ exact seq of $G$-modules s.t. $H^n(G, B^i) = 0\ \forall n \geq 1$ $\forall i = 1, \dots, s$

then $H^n(G, C) = H^{n+s}(G, A)$ $n \geq 1$.

Proof: separate into short exact sequences

$$0 \to B^1 \to C^1 \to 0 \qquad C^1 = \mathrm{Im}(B^1 \to B^2) = \ker(B^2 \to B^3)$$

$$0 \to C^1 \to B^2 \to C^2 \to 0$$

$$\vdots$$

$$0 \to C^{s-1} \to B^s \to C \to 0.$$

$$\Rightarrow H^n(G, C) = H^{n+1}(G, C^{s-1}) = \dots = H^{n+s}(G, A) \quad \square$$

- Consider a cyclic resolution

$$0 \to A \xrightarrow{\varepsilon} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \to \cdots \qquad \text{i.e. } H^n(G, I^i) = 0 \quad \forall i \geq 0, n \geq 1$$

we have exact seq $0 \to A \xrightarrow{\varepsilon} I^0 \to \dots \to I^{n-1} \xrightarrow{d^{n-1}} \mathrm{Im}\, d^{n-1} \to 0$, $\mathrm{Im}\, d^{n-1} = \ker d^n$

$$\Rightarrow H^{n+1}(G, A) = \underbrace{H^1(G, \ker d^n)}_{\text{how to compute ?}} \quad \forall n \geq 0$$

- How to compute $H^1(G, \ker d^n)$ ?

We have $0 \to \ker d^n \to I^n \xrightarrow{d^n} \ker d^{n+1} = \mathrm{im}\, d^n \to 0$

$$\rightsquigarrow 0 \to (\ker d^n)^G \to (I^n)^G \to (\ker d^{n+1})^G$$

$$\to H^1(G, \ker d^n) \to 0 = H^1(G, I^n)$$

$$\Rightarrow H^{n+1}(G, A) = H^1(G, \ker d^n) = \mathrm{coker}\left(d^n : (I^n)^G \to (\ker d^{n+1})^G\right)$$

$$= \frac{(\ker d^{n+1})^G}{\operatorname{Im} d^n((I^n)^G)} = H^{n+1}((I^\cdot)^G)$$

- **Summary**: $H^n(G,A)$ can be computed using acyclic resolutions (e.g. induced resolutions).

For $n=0$ $\quad H^0((I^\cdot)^G) = \ker(d^0 : (I^0)^G \to (I^1)^G) = (\ker d^0)^G$

$$0 \to (I^0)^G \xrightarrow{d^0} (I^1)^G \xrightarrow{d^1} \cdots$$

$= (\operatorname{Im}\varepsilon)^G$

$$0 \to A \xrightarrow{\varepsilon} I^0 \xrightarrow{d^0} I^1 \to \cdots$$

$\simeq A^G = H^0(G,A).$

(*) We use these to do concrete computations.

$H^n(G,A) = \operatorname{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, A)$ can be computed from projective resolution of $\mathbb{Z}$.

For $n \geq 0$. $P_n = \mathbb{Z}[G^{n+1}]$

$G \curvearrowright P_n$ : $\quad g(g_0, \ldots, g_n) = (gg_0, \ldots, gg_n).$

$\rightsquigarrow P_n$ is $G$-module.

$P_n$ is free with basis $\{(1, g_1, \ldots, g_n) : g_1, \ldots, g_n \in G\}$

since $\forall \displaystyle\sum_{g_0,\ldots,g_n} n_{g_0,\ldots,g_n} (g_0, \ldots, g_n) \in P_n$, $\quad n_{g_0,\ldots,g_n} \in \mathbb{Z}$

$$= \sum_{g_0 \in G} g_0 \cdot \sum_{g_1,\ldots,g_n \in G} n_{g_0,\ldots,g_n} (1, g_0^{-1}g_1, \ldots, g_0^{-1}g_n)$$

$\Rightarrow P_n$ is free $\mathbb{Z}[G]$-module with such basis.

- Define $d : P_n \to P_{n-1}$ by

$$d(\underbrace{g_0, \ldots, g_n}) = \sum_{i=0}^{n} (-1)^i (g_0, \ldots, \widehat{g_i}, \ldots, g_n)$$

basis of $P_n$ as abelian group.

Note $d \circ d = 0$ so we have

$$\mathbb{Z} \xleftarrow{\varepsilon} \underbrace{P_0 \xleftarrow{d} P_1 \xleftarrow{d} P_2}_{\text{free } \mathbb{Z}[G]\text{-module}} \xleftarrow{d} \cdots$$

$\varepsilon : \mathbb{Z}[G] \to \mathbb{Z}$ augmented map.

i.e. $\varepsilon(g) = 1 \ \forall g \in G$ i.e. $\varepsilon\left(\sum_{g\in G} n_g g\right) = \sum_{g\in G} n_g$.

- This sequence is exact: $\varepsilon$ is surjective, choose any $h \in G$

In fact, define $k : P_n \to P_{n+1}$ by

$$k(g_0, \dots, g_n) = (h, g_0, \dots, g_n).$$

$\Rightarrow d \circ k + k \circ d = \mathrm{id}\, P_n$. since

$$d(k(g_0, \dots, g_n)) + k(d(g_0, \dots, g_n))$$

$$= d(h, g_0, \dots, g_n) + \sum_{i=0}^{n} (-1)^i (h, g_0, \dots, \hat{g}_i, \dots, g_n) = (g_0, \dots, g_n).$$

If $x \in P_n$, $dx = 0 \Rightarrow x = d(k(x)) \Rightarrow$ exact.

- Thus, $0 \leftarrow \mathbb{Z} \xleftarrow{\varepsilon} P_0 \xleftarrow{d} P_1 \xleftarrow{d} \cdots$ in $\mathrm{Mod}_G$ is free resolution of $\mathbb{Z}$.

$$\Rightarrow 0 \to \mathrm{Hom}_G(P_0, A) \xrightarrow{d^*} \mathrm{Hom}_G(P_1, A) \xrightarrow{d^*} \cdots$$

$$\Rightarrow H^n(G, A) \simeq H^n(\mathrm{Hom}_G(P_\bullet, A)). \text{ (from previous result)}$$

* What is $\mathrm{Hom}_G(P_n, A)$?

$$\mathrm{Hom}_G(P_n, A) = \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^{n+1}], A)$$

$$\mathbb{Z}[G^{n+1}] = \bigoplus_{g_1, \dots, g_n} \mathbb{Z}[G] \cdot (1, g_1, \dots, g_n)$$

$\overset{1-1}{=}$ $\{$maps $\varphi : G^{n+1} \to A$ s.t. $\varphi(gg_0, \dots, gg_n) = g\cdot\varphi(g_0, \dots, g_n)\ \forall g, g_0, \dots, g_n \in G\}$

Def: $\tilde{C}^n(G,A) = \{$maps $\varphi : G^{n+1} \to A$ s.t. $\varphi(gg_0, \dots, gg_n) = g\,\varphi(g_0, \dots, g_n)\}$

$\cong$

$Hom_G(P_n, A)$

homogeneous n-chain (đối dây chuyền thuần nhất bậc n)

$\tilde{d} : \tilde{C}^n(G,A) \to \tilde{C}^{n+1}(G,A)$ is given by

$$(\tilde{d}\,\tilde{\varphi})(g_0, \dots, g_{n+2}) := \sum_{i=0}^{n+1} (-1)^i \varphi(g_0, \dots, \hat{g_i}, \dots, g_{n+1}).$$

$$\Rightarrow H^n(G,A) = \frac{\ker \tilde{d}\,(\tilde{C}^n \to \tilde{C}^{n+1})}{\operatorname{im}\,(\tilde{d} : \tilde{C}^{n-1} \to \tilde{C}^n)}.$$

- Better description:

$\varphi \in \tilde{C}^n(G,A)$ $\quad \varphi : G^{n+1} \to A$

$$\varphi(gg_0, \dots, gg_n) = g\cdot\varphi(g_0, \dots, g_n)$$

$\rightsquigarrow$ $\varphi$ is uniquely determined by its value at $\varphi(1, g_1, g_1g_2, \dots, g_1\cdots g_n)$
$g_1, \dots, g_n \in G$.

$C^n(G,A) := \{$maps $\varphi : G^n \to A\}$ inhomogeneous n-cochains

$d : C^n(G,A) \to C^{n+1}(G,A)$.
$\quad\ \ \varphi^G$

$$(d\varphi)(g_1, \dots, g_{n+1}) = \tilde{d}\tilde{\varphi}(1, g_1, g_1g_2, \dots, g_1\cdots g_{n+1})$$

$$= \tilde{\varphi}(g_1, g_1g_2, \dots, g_1\cdots g_{n+1}) + \sum_{i=1}^{n} (-1)^i (1, g_1, \dots, g_1\cdots g_{i-1}, g_1\cdots g_{i+1}, \dots, g_1\cdots g_{n+1})$$

$$+ (-1)^{n+1}\tilde{\varphi}(1, g_1, \dots, g_1\cdots g_n)$$

$$= g_1\varphi(g_2,\dots,g_{n+1}) + \sum_{i=1}^{n}(-1)^i\varphi(g_1,g_2,\dots,g_{i-1}g_i,\dots,g_{n+1})$$
$$+(-1)^{n+1}\varphi(g_1,\dots,g_n).$$

Denote $Z^n(G,A) = \ker(d: C^n \to C^{n+1})$ n-cocycles

$B^n(G,A) = \operatorname{Im}(d: C^{n-1}, C)$ n-coboundary

$H^n(G,A) = Z^n(G,A)/B^n(G,A)$ nth homology group of G with coef in A.

- $C^0(G,A) = \{\text{maps } G^0 \to A\} = A$
- $C^1(G,A) = \{\text{maps } G \to A\}$
- $C^2(G,A) = \{\text{maps } G\times G \to A\}$

$d: C^0(G,A) \to C^1(G,A)$ $\quad x \in A$, $dx(g) = g\cdot x - x$

$x \in Z^0(G,A) \Leftrightarrow dx = 0 \Leftrightarrow gx = x\ \forall g \Leftrightarrow x \in A^G$

$B^0(G,A) = 0$

$H^0(G,A) = A^G.$

- $d: C^1(G,A) \to C^2(G,A)$ $\quad d\varphi(g,h) = g\cdot\varphi(h) - \varphi(gh) + \varphi(g)$

$Z^1(G,A) = \{\varphi: G \to A \quad \varphi(gh) = g\cdot\varphi(h) + \varphi(g)\}$

$B^1(G,A) = \{\varphi: G \to A \quad \exists x \in A \quad \varphi(g) = g.x - x \quad \forall g \in G\}$

For $\varphi \in Z^1(G,A)$ called cross homomorphism đồng cấu lệch

$\varphi \in B^1(G,A)$ called principal homomorphism

$$H^1(G,A) = \frac{\{\text{cross hom } G \to A\}}{\{\text{principal hom } G \to A\}}.$$

- $\varphi \in Z^2(G,A)$ $0 = d\varphi : G\times G \to A \Rightarrow g\cdot\varphi(h,k) - \varphi(gh,k)$
$+ \varphi(g,hk) - \varphi(g,h) = 0.$

$\varphi \in C^1(G,A)$ $\quad d\varphi(g,h) = g\varphi(h) - \varphi(gh) + \varphi(g).$

$$H^2(G,A) = Z^2(G,A)/B^2(G,A)$$

Classifies extensions of $G$ by $A$ (see other note).

$$1 \to A \xrightarrow{i} E \xrightarrow{p} G \to 1$$

s.t. $i(p(x).a) = x\, i(a)\, x^{-1}$ $\forall a \in A, x \in E$

⊛

$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ exact $G$-modules

$\rightsquigarrow$ $0 \to A^G \to B^G \to H^1(G,A) \xrightarrow{\alpha_*} H^1(G,B) \xrightarrow{\beta_*} H^1(G,C) \to \ldots$

where $H^n(G,A) \to H^n(G,B)$

$$[\varphi : G^n \to A] \mapsto [\alpha\circ\varphi : G^n \to B]$$

and $\delta : H^n(G,C) \to H^{n+1}(G,A)$.

$$\delta[\varphi] = [\psi].$$

$$\begin{tikzcd}
 & \tilde{\varphi} & \varphi \\
C^n(A) \arrow[r, "\alpha_*"] \arrow[d, "d"] & C^n(B) \arrow[r, "\beta_*"] \arrow[d, "d"] & C^n(C) \arrow[d, "d"] \\
C^{n+1}(A) \arrow[r] & C^{n+1}(B) \arrow[r] & C^{n+1}(C) \\
Z^{n+1}(A) \ni \psi \arrow[r, "\alpha_*"] & d\tilde{\varphi} \arrow[r, "\beta_*"] &
\end{tikzcd}$$

where:

$\varphi$ is a $n$-cocycle, $d\varphi = 0$

can take $\tilde{\varphi} : G^n \to B$ so

$\beta\circ\tilde{\varphi} = \varphi$

$\beta\circ d\tilde{\varphi} = d(\beta\circ\tilde{\varphi}) = d\varphi = 0 \Rightarrow d\tilde{\varphi}$ takes value in $\ker\beta = \operatorname{im}\alpha$

$\Rightarrow d\tilde{\varphi} = \alpha\circ\psi$ for some $\psi : G^{n+1} \to A$.

$\alpha\circ d\psi = d(\alpha\circ\psi) = d(d\tilde{\varphi}) = 0 \Rightarrow d\psi = 0$ ($\alpha$ injective).

→ Because we have a good description of $C^n(A)$ just as maps $G^n \to B$ we can choose $\tilde{\varphi}$ to be any map (no homomorphism condition).

## Lecture: 13/07/2020 Group cohomology (continued)

- Property: $H^n(G, \underbrace{\prod_i A_i}) \simeq \prod_i H^n(G, A_i)$

  diagonal action $g \cdot (a_i)_i = (g \cdot a_i)_i$

$$\mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, \prod_i A_i) \simeq \prod_i \mathrm{Ext}^n_{\mathbb{Z}[G]}(G, A_i)$$

- $\alpha : G' \to G$ hom of groups $\quad A$: $G$-module, $A'$: $G'$-module

  $\rightsquigarrow A$ is also $G'$ module $\quad g' \cdot x := \alpha(g') \cdot x$

  Homomorphism $\beta : A \to A'$ of abelian groups

  We say $(\alpha, \beta)$ is compatible if $\beta(\alpha(g) \cdot x) = g \cdot \beta(x) \quad \forall \begin{matrix} g \in G' \\ x \in A \end{matrix}$

$$\Rightarrow \begin{tikzcd} G^n \arrow[r, "\varphi"] & A \arrow[d, "\beta"] \\ G'^n \arrow[u, "\alpha^n"] \arrow[r, "\beta \circ \varphi \circ \alpha^n"'] & A' \end{tikzcd}$$

then $C^n(G, A) \to C^n(G', A')$

$\varphi \mapsto \psi(g_1, \ldots, g_n) := \beta(\varphi(\alpha(g_1), \ldots, \alpha(g_n)))$

This commutes with $d$.

$$\begin{tikzcd} C^n(G, A) \arrow[r] \arrow[d, "d"'] & C^n(G', A') \arrow[d, "d"] \\ C^{n+1}(G, A) \arrow[r] & C^{n+1}(G', A') \end{tikzcd}$$

$\rightsquigarrow$ Induces $H^n(G, A) \to H^n(G', A')$

$[\varphi] \mapsto [(g_1, \ldots, g_n) \mapsto \beta(\varphi(\alpha(g_1), \ldots, \alpha(g_n)))]$

Eg: $H \leq G \quad H \overset{\alpha}{\hookrightarrow} G$

$A$ is $G$-module $\overset{\beta}{\rightsquigarrow}$ $H$-module

$\rightsquigarrow H^n(G, A) \to H^n(H, A)$

$$[\varphi] \longmapsto [(h_1, \dots, h_n) \mapsto \varphi(h_1, \dots, h_n)]$$

restriction morphism

Eg2: $H \leq G$, $A$: $H$-module

$H \overset{\alpha}{\hookrightarrow} G$ $\quad \mathrm{Ind}_H^G A$ : $G$-module

$\mathrm{Ind}_H^G \xrightarrow{\beta} A$

$(\varphi: G \to A) \mapsto \varphi(1_G)$

$\rightsquigarrow H^n(G, \mathrm{Ind}_H^G A) \xrightarrow{\sim} H^n(H, A)$

This is the isomorphism constructed in Shapiro's lemma

$$[\varphi] \longmapsto [(h_1, \dots, h_n) \mapsto \varphi(h_1, \dots, h_n)(1_G)].$$

- Restriction morphism $\quad \mathrm{Res}: H^n(G, A) \longrightarrow H^n(H, A)$

$$[\varphi] \longmapsto [\varphi|_{H^n}]$$

can be also describe as follows:

$A \hookrightarrow \mathrm{Ind}_H^G A \quad x \mapsto (g \mapsto g.x)$ induces

$$\underbrace{H^n(G, A) \to H^n(G, \mathrm{Ind}_H^G A) \simeq H^n(H, A)}_{\mathrm{Res}}.$$

- Eg3: $H \trianglelefteq G$ normal subgroup. $\quad \alpha: G \to G/H$.

$A$: $G$-module $\quad A^H$: $G/H$-module $\quad gH.x = g.x$

then we have $\beta: A^H \hookrightarrow A$. $\quad$ Inflation morphism (lạm

As $\alpha, \beta$ compatible so $\quad H^n(G/H, A^H) \to H^n(G, A)$ $\quad$ phát)

$$[\varphi] \longmapsto [(g_1, \dots, g_n) \mapsto \varphi(g_1H, \dots, g_nH)]$$

- Eg4: Fix $g_0 \in G$, $\alpha: G \to G$ $\quad g \mapsto g_0 g g_0^{-1}$

$A$- $G$ module $\quad \beta: A \to A \quad \beta(x) = g_0^{-1}.x$.

$$\rightsquigarrow H^n(G, A) \xrightarrow{\simeq} H^n(G, A)$$

$n=0$ $\quad A^G \xrightarrow{=} A^G$

$n\geq 1$ $\quad B = \mathrm{Ind}^G A$ , $C = B/A$

exact $0 \to A \to B \to C \to 0$

$\rightsquigarrow$ Long exact seq:

$n\geq 1$

$H^n(G,B)=0$

B induced module

$$\begin{tikzcd} \cdots \arrow[r] & H^{n-1}(G,B) \arrow[r] \arrow[d, equal, "\text{induction}"'] & H^{n-1}(G,C) \arrow[r, "\delta"] \arrow[d, Rightarrow] & H^n(G,A) \arrow[r] \arrow[d, equal] & 0 \\ \cdots & H^{n-1}(G,B) \arrow[r] & H^{n-1}(G,C) \arrow[r, "\delta"] & H^n(G,A) \arrow[r] & 0 \end{tikzcd}$$

$\square$

**Inf-Res exact sequence:**

$H \trianglelefteq G$, $A$ $G$-module $n\geq 1$ s.t. $H^i(H,A)=0$ $\forall 0<i<n$

then we have exact seq:

$$0 \to H^n(G/H, A^H) \xrightarrow{\mathrm{Inf}} H^n(G,A) \xrightarrow{\mathrm{Res}} H^n(H,A) \dashrightarrow$$

can be longer

- If $n=1$: the condition $H^i(H,A)=0$ $\forall 0<i<n$ is vacuous

$$0 \to H^1(G/H, A^H) \xrightarrow[\mathrm{Inf}]{} H^1(G,A) \xrightarrow{\mathrm{Res}} H^1(H,A) \quad \text{exact}$$

- Inf injective: $\varphi: G/H \to A^H$ 1 cocycle
  s.t. $\mathrm{Inf}(\varphi) = \psi: G \to A$ is 1-coboundary

$\Rightarrow \exists a\in A$ $\quad \psi(g) = g.a - a$ $\quad \forall g\in G$.

$\qquad\qquad \varphi(gH) = g.a - a \quad \forall g \in G$.

$g = 1_G \Rightarrow \varphi(1_G H) = 0 \Rightarrow \varphi(hH) = 0 \ \forall h\in H$.

$\qquad \Rightarrow h.a = a \Rightarrow a \in A^H$.

$\quad \Rightarrow \varphi(gH) = gH.a - a \quad \forall a \in G \Rightarrow \varphi$ 1-coboundary

- Show: $\mathrm{Res}\circ\mathrm{Inf} = 0$.

$\varphi \in Z^1(G/H, A^H)$, $\mathrm{Inf}(\varphi) = \psi \in Z^1(G,A)$

$\qquad \psi(g) := \varphi(g.H)$.

$\varphi(1_{G/H}) = 0 \Rightarrow \varphi(h) = 0 \ \forall h \in H.$

$\Rightarrow \mathrm{Res}[\varphi] = [\varphi|_H] = 0$

$\Rightarrow \mathrm{Res} \circ \mathrm{Inf} = 0.$

- Show $\mathrm{Ker}(\mathrm{Res}) \subseteq \mathrm{Im}(\mathrm{Inf})$

$$0 \to H^1(G/H, A^H) \xrightarrow{\mathrm{Inf}} H^1(G,A) \xrightarrow{\mathrm{Res}} H^1(H,A)$$

If $\varphi \in H^1(G,A)$ so $\mathrm{Res}(\varphi) = 0$ i.e. $\varphi|_H$ is 1-coboundary

$\Rightarrow \exists a \in A : \varphi(h) = ha - a \ \ \forall h \in H.$

Define $\varphi' : G \to A$ $\quad \varphi'(g) = \varphi(g) - g.a + a.$

$\Rightarrow \varphi' \in Z^1(G,A), \ \varphi'(h) = 0 \ \forall h \in H.$

Define $\psi : G/H \to A^H$ by $\psi(gH) = \varphi'(g) = \varphi(g) - g.a + a$

This is well-defined: Let $h \in H$, $\varphi'(gh) = \varphi(gh) - gha + a$

$= g.\varphi(h) + \varphi(g) - gha + a$ ($\varphi$ is cocycle)

$= g\underbrace{(\varphi(h) - h.a + a)}_{=0} + \underbrace{\varphi(g) - g.a + a}_{\varphi'(g)}$

Also $\psi : G/H \to A^H$ is cocycle and $\mathrm{Inf}[\psi] = [\varphi'] = [\varphi]$ $\square$

- If $n > 1$: $\quad 0 \to A \to B \to C \to 0 \quad B = \mathrm{Ind}^G A$, $C = B/A$

(kĩ thuật nhảy chiều) $\quad$ Since $B$ is induced, hence (B is induced as H-mod)

acyclic $\Rightarrow H^n(H,C) = H^{n+1}(H,A) \ \ \forall n > 0$

$\Rightarrow \forall 0 < i < n-1 \quad H^i(H,C) = H^{i+1}(H,A) = 0$ assumption

By induction, the top row is exact.

$$\begin{tikzcd} 0 \arrow[r] & H^{n-1}(G/H, C^H) \arrow[r, "\mathrm{Inf}"'] \arrow[d, phantom, "\wr"] & H^{n-1}(G,C) \arrow[r, "\mathrm{Res}"'] \arrow[d, phantom, "\wr"] & H^{n-1}(H,C) \arrow[d, phantom, "\wr"] \\ 0 \arrow[r] & H^n(G/H, A^H) \arrow[r, "\mathrm{Inf}"'] & H^n(G,A) \arrow[r, "\mathrm{Res}"'] & H^n(H,A) \end{tikzcd}$$

Since Inf, Res is functorial (diagram commutes). So the below row is also exact $\square$

(✱) If $H \leq G$ has finite index $G = \bigsqcup_{s \in H} sH$

$A$: $G$-module

$x \in A^H$: Chuẩn/norm $\mathrm{Nm}_{G/H} : A^H \to A^G$

$x \mapsto \sum_{g \in S} g.x$

$s.hx = sx \ \forall h \in H$

$\to H^0(H,A) = A^H \to H^0(G,A) = A^G$

We want to extend this $H^n(H,A) \to H^n(G,A)$ ?

co restriction morphism cor

- $\mathrm{Ind}_H^G A \to A \quad \varphi \mapsto \sum_{s \in S} s \cdot \varphi(s^{-1})$
  - it is well-defined: $\forall h \in H \quad sh \cdot \varphi(h^{-1}s^{-1}) = s\,\varphi(hh^{-1}s^{-1}) = s\varphi(s^{-1})$.

$$\begin{tikzcd} H^n(H,A) \arrow[r, "\sim"', "\text{shapiro}"'] \arrow[rr, bend right, "\mathrm{Cor}"'] & H^n(G, \mathrm{Ind}_H^G A) \arrow[r] & H^n(G,A) \end{tikzcd}$$

Cor cores. triction

$\varphi : H^n \to A$ cocycle $\longmapsto \mathrm{Cor}[\varphi] = \left[(g_1, \ldots, g_n) \mapsto \sum_{s \in S} s \cdot \varphi(s^{-1}g_1y_1, \ldots, s^{-1}g_ny_n)\right]$

where $y_i \in G$ s.t. $sH = g_iy_iH$.

- Fact: $$\begin{tikzcd} H^n(G,A) \arrow[r, "\mathrm{Res}"] \arrow[rr, bend right, "{[G:H]\times}"'] & H^n(H,A) \arrow[r, "\mathrm{Cor}"] & H^n(G,A) \end{tikzcd}$$

Proof: $$\begin{tikzcd} H^n(G,A) \arrow[r] \arrow[rr, bend right, "\mathrm{Res}"'] & H^n(G, \mathrm{Ind}_H^G A) \arrow[r, "\sim"] \arrow[rr, bend right, dashed] & H^n(H,A) \arrow[r, "\mathrm{Cor}"] & H^n(G,A) \end{tikzcd}$$

$A \to \mathrm{Ind}_H^G A \longrightarrow A$

$x \mapsto \varphi_x : G \to A \longmapsto \sum_{s \in S} s \cdot \varphi_x(s^{-1}) = \sum_{s \in S} s.s^{-1}x = [G:H].x$

$g \mapsto g.x$

□

- Corollary 1: $|G| = m$. $mH^n(G,A) = 0 \ \forall n \geq 1$

Consider
$$\begin{tikzcd} H^n(G,A) \arrow[r, "\text{Res}"] \arrow[rr, bend right, "m\times"'] & H^n(\{1\}, A) \arrow[r, "\text{Cor}"] & H^n(G,A) \end{tikzcd}$$
$H^n(\{1\},A) = 0$

$mx = 0$.

- Corollary 2: $G$ finite, $A$ finitely generated (hữu hạn sinh) abelian group $\Rightarrow H^n(G,A)$ finite.

Indeed, if $a_1, \dots, a_k$ generates $A$. $\Rightarrow H^n(G,A)$ finitely generated and torsion

$\Rightarrow H^n(G,A)$ finite.

- Corollary 3: $G$ finite, $G_p$ is $p$-Sylow subgroup of $G$ ($p \nmid [G:G_p]$) then $\text{Res}: H^n[G,A) \to H^n(G_p,A)$ is injective on $H^n(G,A)[p^\infty]$

$A[p^\infty] = \bigcup_{n=0}^{\infty} A[p^n]$

$A[m] = \{x \in A, mx = 0\}$ ($p$-primitive part of $A$)

Proof: $\text{Cor} \circ \text{Res}: H^n(G,A) \to H^n(G,A)$ is multiplication by $[G:G_p]$
$$\begin{tikzcd} H^n(G,A) \arrow[r] \arrow[dr] & H^n(G,A) \\ & H^n(G_p,A) \arrow[u] \end{tikzcd}$$

$\Rightarrow$ It is injective as $p \nmid [G:G_p] \mid H^n(G,A)[p^\infty]$

$\Rightarrow$ Res is injective $\square$

## ⊛ Cohomology of cyclic groups

$$C_n = \langle \sigma \mid \sigma^n = 1 \rangle \simeq \mathbb{Z}/n\mathbb{Z} \qquad C_\infty = \langle \sigma \rangle \simeq \mathbb{Z}$$

$$\mathbb{Z}[C_n] = \left\{ \sum_{i=0}^{n-1} n_i \sigma^i \quad \sigma^n = 1 \right\} = \mathbb{Z}[\sigma] / \langle \sigma^n - 1 \rangle$$

We find
$$0 \leftarrow \mathbb{Z} \xleftarrow{\varepsilon} \mathbb{Z}[C_n] \xleftarrow{\times(\sigma-1)} \mathbb{Z}[C_n] \xleftarrow{\times(\sigma^{n-1}+\cdots+\sigma+1)} \mathbb{Z}[C_n] \xleftarrow{\times(\sigma-1)} \cdots$$
$$\sum_{i=0}^{n-1} n_i \leftarrow\!\shortmid \sum_{i=0}^{n-1} n_i \sigma_i$$

This is exact seq as $(\sigma-1)(\sigma^{n-1}+\cdots+1) = \sigma^n - 1 = 0$.

This is a free resolution of $\mathbb{Z}$.

$$0 \leftarrow \mathbb{Z} \xleftarrow[\varepsilon]{} \mathbb{Z}[C_n] \xleftarrow[\times(\sigma-1)]{} \mathbb{Z}[C_n] \xleftarrow[\times(\sigma^{n-1}+\cdots+1)]{} \mathbb{Z}[C_n] \xleftarrow[\times(\sigma-1)]{} \cdots$$

$\downarrow \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \cdot)$

$$0 \to A \xrightarrow{(\sigma-1)} A \xrightarrow{N_G} A \longrightarrow \cdots$$
$$a \mapsto \sigma a - a$$
$$a \overset{N_G}{\longmapsto} \sigma^{n-1} a + \cdots + a = N_G(a)$$

$$\Rightarrow H^0(G, A) = \ker(\sigma - 1)$$
$$H^n(G, A) = \ker N_G / \mathrm{Im}(\sigma-1) \quad n \text{ odd}$$
$$H^n(G, A) = \ker(\sigma-1) / \mathrm{Im}\, N_G \quad n \text{ even} > 0$$

- When $G = C_\infty = \langle \sigma \rangle$ then our seq is $\quad (\mathbb{Z}[G] = \mathbb{Z}[\sigma, \sigma^{-1}] \simeq \mathbb{Z}[x,y]/(xy-1))$
$$0 \leftarrow \mathbb{Z} \xleftarrow{\varepsilon} \mathbb{Z}[G] \xleftarrow{(\sigma-1)} \mathbb{Z}[G] \leftarrow 0.$$
a free resolution.

$$\rightsquigarrow 0 \to A \to A \to 0 \Rightarrow \begin{array}{l} H^0(G, A) = A^\sigma = \ker(\sigma-1) \\ H^1(G, A) = A / \langle \sigma a - a, a \in A \rangle = A_G \end{array}$$
$$a \mapsto \sigma a - a$$

$$H^n(G, A) = 0 \quad n \geq 2. \quad \text{Coinvariant}$$

⊛ **Theorem:** $G$ finite, $H^1(H,A)=H^2(H,A)=0$ $A$ $G$ module
$\forall$ subgroup $H\leq G$

Then $H^n(G,A)=0$ $\forall n\geq 1$.

Proof: Induction on $|G|$ order = cap

- If $G$ cyclic $\Rightarrow$ ok.
- If $G$ is solvable (i.e. $\exists$ $G=G_0\trianglerighteq G_1\trianglerighteq\cdots\trianglerighteq G_n=\{1\}$
$G_i/G_{i+1}$ abelian)

take $H\trianglelefteq G$ so $G/H$ cyclic

By induction $H^n(H,A)=0$ $\forall n\geq 1$.

Consider Inf-Res seq:

$$0\to H^n(G/H,A^H)\xrightarrow[\sim]{\text{Inf}} H^n(G,A)\xrightarrow{\text{Res}} H^n(H,A)=0.$$

$$\Rightarrow H^n(G,A)\simeq H^n(G/H,A^H).$$

We have $0=H^1(G,A)\simeq H^1(G/H,A^H)$

$0=H^2(G,A)\simeq H^2(G/H,A^H)$

Since $G/H$ cyclic $\Rightarrow H^n(G/H,A^H)=0=H^n(G,A)$.

- If $G$ arbitrary. $p$ prime, $G_p$ is $p$-Sylow subgroup of $G$ $\Rightarrow$ $G_p$ is solvable

$\Rightarrow H^n(G_p,A)=0$ $\forall n\geq 1$

But Res: $H^n(G,A)[p^\infty]\to H^n(G_p,A)$ injective

$\Rightarrow \underbrace{H^n(G,A)}_{}[p^\infty]=0$ $\forall p$.

torsion since $G$ finite $\Rightarrow H^n(G,A)=\bigoplus_{p\text{ prime}} H^n(G,A)[p^\infty]$
$=0$ $\square$

⊛ **Tate's theorem:** $G$ finite group

$A$ is $G$-module s.t. $\forall H \leq G$, we have $H^1(H,A)=0$ and $H^2(H,A)$ is cyclic of order $|H|$.

Then there are isomorphisms $H^n(G, \mathbb{Z}) \xrightarrow{\sim} H^{n+2}(G,A)$ $\forall n \geq 1$

trivial action.

Proof: $H^2(G,A)$ cyclic order $|G|$

Choose generator $\gamma$ for $H^2(G,A)$

$\forall H \leq G$

$$\begin{tikzcd} H^2(G,A) \arrow[r, "Res"] \arrow[rr, bend right, "{[G:H] = |G|/|H|}"'] & H^2(H,A) \arrow[r, "Cor"] & H^2(G,A) \end{tikzcd}$$

$\gamma$ $\qquad\qquad \frac{|G|}{|H|}\gamma$ order $|H|$

$\Rightarrow$ Res $\gamma$ generates $H^2(H,A)$ (since Res, Cor group hom and $|G|/|H| \, \gamma$ order $|H|$ and $H^2(H,A)$ order $|H|$)

- Take $\gamma = [\varphi]$ ; $\varphi : G \times G \to A$

$$\left.\begin{aligned} \varphi(1,g) &= \varphi(1,1) \quad \forall g \in G \\ \varphi(g,1) &= g\,\varphi(1,1) \quad \forall g \in G \end{aligned}\right\}$$

Since $\varphi$ 2-cocycle so $g \cdot \varphi(h,k) - \varphi(gh,k) + \varphi(g,hk) + \varphi(g,h) = 0$.

Let $A(\varphi) := A \oplus \bigoplus_{\substack{g \in G \\ g \neq 1_G}} \mathbb{Z} x_g$ splitting module of $\varphi$

$G \curvearrowright A(\varphi)$ by $g \cdot x_h := x_{gh} - x_g + \varphi(g,h)$.

$x_1$ is understood as $\varphi(1,1) \in A$

$(h = 1 : g \cdot \varphi(1,1) = \varphi(g,1))$.

It is group action since

$$g(g'x_h) = g\cdot x_{g'h} - g\cdot x_{g'} + g\cdot\varphi(g',h)$$
$$= x_{gg'h} - x_g + \varphi(g,g'h) - x_{gg'} + x_g - \varphi(g,g')$$
$$+\, g\cdot\varphi(g',h)$$
$$= x_{gg'h} - x_{gg'} + \varphi(gg',h) = gg'\cdot x_h.$$
$$1\cdot x_h = x_h - x_1 + \varphi(1,h) = 0.$$

$\Rightarrow \varphi(g,h) = g\cdot x_h - x_{gh} + x_g$ is 2-coboundary in $A(\varphi)$

$$\underbrace{G\times G \xrightarrow{\varphi} A \hookrightarrow A(\varphi)}_{2\text{ coboundary}}$$

- We prove that $H^1(H, A(\varphi)) = H^2(H, A(\varphi)) = 0$ $\forall H \leq G$.

$$0 \to \underset{\mathbb{Z}\langle g-1,\, g\neq 1\rangle}{I_G} \to \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \quad \text{exact}$$
$$g \mapsto 1$$

$\mathbb{Z}[G] \simeq \mathrm{Ind}^G \mathbb{Z}$ is induced so $H^n(G, \mathbb{Z}[G]) = 0$ $\forall n \geq 1$

$\mathrm{Ind}^G \mathbb{Z} \simeq \text{maps}(G, \mathbb{Z})$

$$\rightsquigarrow\ 0 \to I_G^H \to \mathbb{Z}[G]^H \xrightarrow{\varepsilon} \mathbb{Z} \to H^1(H, I_G) \to 0$$

$$\Rightarrow H^1(H, I_G) = \mathbb{Z}/\mathrm{Im}\,\varepsilon = \mathbb{Z}/|H|\mathbb{Z}$$

$\varepsilon: \mathbb{Z}[G]^H \to \mathbb{Z}$ : $\sum_{g\in G} n_g g$ H-invariant $\Rightarrow n_{gh} = n_g$ $\forall g\in G$, $h\in H$

$\Rightarrow$ if $G = \bigsqcup_{s\in S} sH$ then $\sum_{g\in G} n_g = |H| \sum_{s\in S} n_s \Rightarrow \mathrm{Im}\,\varepsilon = |H|\mathbb{Z}$.

$$\Rightarrow H^1(H, I_G) = \mathbb{Z}/|H|\mathbb{Z}.$$

- From $0 \to I_G \to \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \to 0$

as $\mathbb{Z}[G]$ acyclic so $H^1(H, \mathbb{Z}) = H^2(H, I_G)$.

$$Z^1(H,\mathbb{Z}) = \{\varphi : H \to \mathbb{Z} \,;\, \varphi(h_1h_2) = \varphi(h_2) + \varphi(h_1)\}$$
$$= \mathrm{Hom}(H,\mathbb{Z}) = 0 \quad \text{since } H \text{ finite}$$
$$\Rightarrow H^2(H, I_G) = H^1(H,\mathbb{Z}) = 0.$$

- Define $\alpha : A(\varphi) \to \mathbb{Z}[G]$ $\quad \alpha(a) = 0 \ \forall a \in A$

$\quad A(\varphi) = A \oplus \bigoplus_{g\neq 1} \mathbb{Z}x_g \qquad \alpha(x_g) = g-1$

$\qquad \Rightarrow \alpha(A(\varphi)) = I_G.$

$\rightsquigarrow$ Exact seq $0 \to A \hookrightarrow A(\varphi) \to I_G \to 0$

$\rightsquigarrow$ Long exact seq of cohomology with coef

$$0 = H^1(H,A) \to H^1(H,A(\varphi)) \to H^1(H,I_G) \ \ (\mathbb{Z}/|H|\mathbb{Z})$$
$$(\mathbb{Z}/|H|\mathbb{Z} \text{ assumption}) \ H^2(H,A) \xrightarrow{0} H^2(H,A(\varphi)) \to H^2(H,I_G) = 0$$

Recall $\gamma \in H^2(G,A)$ generator

$\quad \mathrm{Res}(\gamma) \in H^2(H,A)$ generator $\quad \gamma : G\times G \to A(\varphi)$ 2-coboundary

$\mathrm{Res}(\gamma) \mapsto 0$ in $H^2(H,A(\varphi))$ $\Rightarrow$ $H^2(H,A) \xrightarrow{0} H^2(H,A(\varphi))$.

$\Rightarrow H^2(H,A(\varphi)) = 0.$ since — surjective.

$\Rightarrow H^1(H,I_G) \to H^2(H,A)$ surjective, both are $\mathbb{Z}/|H|\mathbb{Z}$

so it is isomorphism

$$\Rightarrow H^1(H,A(\varphi)) = \ker(H^1(H,I_G) \to H^2(H,A)) = 0$$

Thus, $H^1(H,A(\varphi)) = H^2(H,A(\varphi))$ $\forall H \leq G$ $\forall n \geq 1$

$\Rightarrow$ By previous theorem $H^n(G,A(\varphi)) = 0 \ \forall n \geq 1.$

We have

$$0 \to A \to \overbrace{A(e)}^{\text{acyclic}} \xrightarrow{\alpha} I_G \to 0$$

$$0 \to I_G \to \underbrace{\mathbb{Z}[G]}_{\text{acyclic}} \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

so : $n \geq 1$.

$\forall n \geq 1$, $\hat{H}^n(G, \mathbb{Z}) \simeq \hat{H}^{n+1}(G, I_G) \simeq \hat{H}^{n+2}(G, A)$ $\square$