Geometric Quantization and Rep theory

- Main thm : Lie group G (semisimple or nilpotent)

  ⇝ adjoint rep $G \curvearrowright \mathfrak{g}$ → Lie alg

  ⇝ co-adjoint $G \curvearrowright \mathfrak{g}^*$

  Irreducible rep of G $\overset{\sim}{\longleftrightarrow}$ — orbits of co-adjoint action

  Thm 7.1.1 in the notes

  by ~~Kirillov.~~

  character = Fourier transform over the corresponding orbit.

- The theorem is an application of geometric quantisation.

  Q: Why this bijection is nice?

Describe explicitly the above bijection for $G = SO_2(\mathbb{R})$ (§1 of Venkatesh's notes)

⊛ Irreducible reps of $SO_3(\mathbb{R})$

- $SO_3(\mathbb{R}) = \{A \in GL_3(\mathbb{R})$ s.t. $AA^T = A^TA = I$, $\det(A) = 1\}$

  $\hookrightarrow$ describe all rotations in $\mathbb{R}^3$.

- P_n = $\mathbb{R}$-vector space of homogeneous pols of deg n in 3 variables

  e.g. $\underbrace{x^2y + yz^2}_{\in P_3}$, $xyz \in P_3$.

$\underline{SO_3} < GL_3 \circlearrowright P_n$ by acting on x, y, z

$\hookrightarrow g(x^2+y^2+z^2) = x^2+y^2+z^2 = r^2$ $\forall g \in SO_3$

$\Rightarrow$ have $SO_3(\mathbb{R})$-equivariant map

$\times r^2 : P_{n-2} \longrightarrow P_n$

- $SO_3(\mathbb{R})$-equivariant map $P_n \to P_{n-2}$
by applying the Laplacian $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$

(by chain rule, $f \in P_n$, $g \in SO_3$
want to show $\Delta(f \circ g) = g \circ \Delta(f)$
$= \Delta g \circ \Delta f$)

- Interestingly, $P = \bigoplus_n P_n$
then xr^2 and Δ define a Lie alg
action of $\mathfrak{sl}_2$ on P. $\rightsquigarrow$ actually a
$h = [xr^2, \Delta] \ldots$ more general fact
will be discussed later.

- All irreps of $SO_3(\mathbb{R})$ are
$V_n := \ker(\Delta : P_n \to P_{n-2})$
$\cong P_n / r^2 P_{n-2}$.
$\hookrightarrow P_n = V_n \oplus r^2 P_{n-2}$.
$\hookrightarrow$ dropbox comment...

Fact: V_n's are all the irreducible rep of $SO_3(\mathbb{R})$, called spherical harmonics

Eg: $V_2 = \mathrm{span}_{\mathbb{R}}(xy, xz, yz, x^2-y^2, y^2-z^2)$.

$$(\partial_x^2 + \partial_y^2 + \partial_z^2)(x^2-y^2) = 2-2 = 0.$$

Rmk: V_n are eigenspaces for the Riemanian Laplacian $\circlearrowright L^2(S^2) =$ square-int func on S^2

$\hookrightarrow$ roughly Δ restricts on S^2.

(dropbox comment).

⊛ Characters of V_n's:

- Recall: $G \xrightarrow{\rho} GL(V)$ finite dim then character $Ch: g \mapsto tr(\rho(g))$.

- $P_n = V_n \oplus r^2 P_{n-2}$, since trace is additive st $Tr_{P_n}(g) = Tr_{V_n}(g) + Tr_{r^2 P_{n-2}}(g)$

$\Rightarrow Ch_{V_n} = Ch_{P_n} - Ch_{P_{n-2}}$ $\qquad Tr_{r^2 P_{n-2}}(g) = Tr_{P_{n-2}}(g)$.

- Any $g \in SO_3(\mathbb{R})$ is conjugate to a rotation around z-axis at angle θ

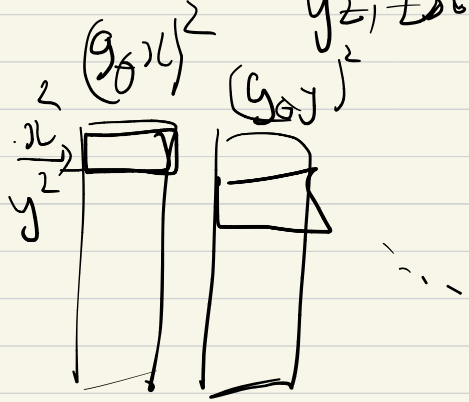
$$g_\theta = \begin{array}{c|ccc} & x & y & z \\ \hline x & \cos\theta & -\sin\theta & 0 \\ y & \sin\theta & \cos\theta & 0 \\ z & 0 & 0 & 1 \end{array}$$

Claim : $ch_{V_n}(g_\theta) = e^{in\theta} + e^{i(n-1)\theta} + \dots + e^{-in\theta}$.

Check for V_2 : ch_{P_2} ? $P_2 = \{x^2, y^2, z^2, xy, yz, zx\}$

Compute trace

$$[x^2]\,(g_\theta x)^2$$



$$\parallel$$

- $[x^2]\,(x\cos\theta + y\sin\theta)^2$
  $= \cos^2\theta$.
- $[y^2](g_\theta y)^2 = \cos^2\theta$.
- $[z^2]\,(g_\theta \underset{z}{z})^2 = 1$

- $[xy]\,(g_\theta x)(g_\theta y)$
  $= [xy]\,(x\cos\theta + y\sin\theta)(-x\sin\theta + y\cos\theta)$
  $= \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

- $[yz](g_\theta y)(g_\theta \underset{\parallel}{z}) = \cos\theta$

$[z\,?]\ \circ \qquad \underset{z}{} = \cos\theta$

$$\mathrm{ch}_{P_2}(g_\theta) = 4\cos^2\theta + 2\cos\theta$$

$$\mathrm{ch}_{P_0}(g_\theta) = 1$$

$P_0 \cong \mathbb{R}$

$$\rightsquigarrow \mathrm{ch}_{V_2}(g_\theta) = 4\cos^2\theta + 2\cos\theta - 1$$

$$= 2(\cos 2\theta + 1) + 2\cos\theta - 1$$

$$= 2\cos 2\theta + 2\cos\theta + 1$$

$$= e^{2i\theta} + e^{-2i\theta} + e^{i\theta} + e^{-i\theta} + e^{0i\theta}. \quad \blacksquare$$

- $\chi_n(g_\theta) = e^{ni\theta} + e^{i(n-1)\theta} + \dots + e^{-ni\theta}$

$$= \frac{e^{i(n+1)\theta} - e^{-in\theta}}{e^{i\theta} - 1}$$

$$= \left(e^{i(n+\frac{1}{2})\theta} - e^{-i(n+\frac{1}{2})\theta}\right) \Big/ \left(e^{i\theta/2} - e^{-i\theta/2}\right).$$

⊕ Show V_n's are all irreps of $SO_3(\mathbb{R})$.
by using χ_n ...

- Recall: G finite group $\rightsquigarrow$ G-invariant inner $\langle , \rangle$ product on $\mathrm{Fun}(G, \mathbb{C}) \supset$ Class functions $\to L^2(G)$

Irre characters form an orthonormal basis wrt $\langle , \rangle$ on span of class functions

$$\chi \text{ irred} \iff \langle \chi, \chi \rangle = 1$$

- G-invariant inner product $\langle , \rangle$ on $L^2(G)$ comes from Haar measure μ on G

i.e. $$\langle f, g \rangle = \int_G f(x)\overline{g(x)}\, d\mu(x)$$

- What is μ for $SO_3(\mathbb{R})$?

• Recall Ramiro's class: Define bi-invariant metric g on $T_e SO_3(\mathbb{R}) = \mathfrak{so}_3$.
↪ killing form on $\mathfrak{so}_3$. $\left\{ \frac{\partial}{\partial J_1}, \dots, \frac{\partial}{\partial J_3} \right\}$

Find 3-dim G-invariant diff form

$$\sqrt{\det(g_{..})}\, dJ_x \wedge dJ_y \wedge dJ_z$$

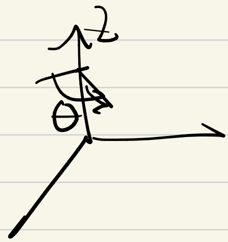
* Local coordinates on $SO_3(\mathbb{R})$;

every element in $SO_3(\mathbb{R})$ corresponds to a rotation, which is uniquely determined by a rotation axis and a rotation angle α

$\Uparrow$

use $\swarrow$ S^2

spherical coordinate (θ, φ)



To represent a matrix in $SO_3(\mathbb{R})$ using $(\theta, \varphi, \alpha)$ $\rightsquigarrow$ use exponential map

$$so_3 \to SO_3(\mathbb{R})$$

<u>To do</u>: Write down ω explicitly ...

<u>Check</u> : χ_n's are orthonormal

$\Rightarrow$ V_n's are all irreducible reps of $SO_3(\mathbb{R})$

— Need to show χ_n's is dense

in the space of class functions

$\subset L^2(SO_3(\mathbb{R}))$. $\blacksquare$