Week 2

Last week we saw characters of irreps of $SO_3(\mathbb{R})$, the V_n, these were

$$\chi_n(g_\theta) = \frac{e^{i(n+\frac{1}{2})\theta} - e^{-i(n+\frac{1}{2})\theta}}{e^{i\theta/2} - e^{-i\theta/2}}$$

Kirrilov's formula

Reformulation:

" $\chi = \left(\begin{array}{c} \text{Fourier transf. of a} \\ \text{surface measure of a} \\ \text{sphere in } \mathbb{R}^3 \end{array} \right)$ "

Recall: $\mathfrak{so}_3(\mathbb{R}) = \{ A \in Mat_3(\mathbb{R}) \mid A^T = -A \}$

and we have a convenient basis

$$J_x = \left(\begin{array}{c|cc} 0 & & 0 \\ \hline 0 & 0 & 1 \\ & -1 & 0 \end{array}\right), \quad J_y = \begin{pmatrix} 0 & 0 & 1 \\ \hline & 0 & \\ \hline -1 & 0 & 0 \end{pmatrix},$$

$$J_z = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & \\ \hline & 0 & 0 \end{array}\right).$$

$$X \in \mathfrak{so}_3 = x J_x + y J_y + z J_z ,$$

"vector" $\vec{J} := (J_x, J_y, J_z)$.

$$X = \vec{x} \cdot \vec{J}.$$

Aside: <u>"Hat map"</u>

$$\tau : \mathfrak{so}_3 \to \mathbb{R}^3$$

$$\vec{x} \cdot \vec{J} \longmapsto \vec{x}$$

which an isomorphism of

the Ad rep. and tautological rep. (rotation action)

$$\boxed{\text{Conj. by a rotation } R \in SO_3(\mathbb{R}) \text{ is geometrically the same rotation in } \mathbb{R}^3.}$$

$$R(\alpha, \beta, \gamma) = R_z(\alpha)\, R_x(\beta)\, R_z(\gamma)$$

Euler angles

$$R_z(\alpha)\, J_x\, R_z(\alpha)^{-1} = \cos\alpha\, J_x - \sin\alpha\, J_y$$

$$(1, 0, 0) \longmapsto (\cos\alpha, -\sin\alpha, 0)$$

* Why?

Recall

$$g \in SO_3 \quad \text{conj.} \quad g_\theta = R_z(\theta)$$
$$\shortparallel$$
$$e^{r\vec{n}\cdot\vec{J}}$$

Claim: $r = \theta$

Facts: $R_{\vec{n}}(\alpha) = e^{\alpha \vec{n}\cdot\vec{J}}$

$\vec{n}$: unit vector in $\mathbb{R}^3$

hom

$$\begin{tikzcd} G \arrow[r, "\varphi"] & G \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "d\varphi|_e"'] \arrow[ur, phantom, "\circlearrowright"] & \mathfrak{g} \arrow[u, "\exp"'] \end{tikzcd}$$

Pf. Denoting the conjugation by $h \in SO_3$ by c_h: $c_h(g) := hgh^{-1}$

For some h,

$$c_h(e^{r\vec{n}\cdot\vec{J}}) = e^{r\,\mathrm{Ad}_h(\vec{n}\cdot\vec{J})}$$
$$\shortparallel$$
$$g$$

$$= g_\theta = e^{\theta J_z}$$

$$\Rightarrow \quad \left| r \, \mathrm{Ad}_n (\vec{n} \cdot \vec{J}) \right|^2 = \left| \theta J_z \right|^2$$

$$r^2 |\underset{\sim}{n}|^2 = r^2 \qquad\qquad \theta^2$$

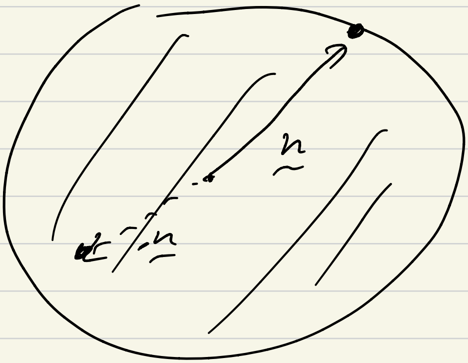
1-1 : the norm induced on so_3 by $\tau : so_3 \to \mathbb{R}^3$

$$\Rightarrow r = \theta .$$

$$\exp : so_3 \to SO_3$$

$$\exp^{-1}(R_{\underset{\sim}{n}}(\alpha)) = \left\{ \alpha \underset{\sim}{n} \cdot \underset{\sim}{J} , \ (\pi + \alpha)(-\underset{\sim}{n}) \cdot \underset{\sim}{J} \right\}$$

$$SO_3 \simeq B_1(\underset{\sim}{0})/\sim$$



$$\sim : \quad \underset{\sim}{n} \sim -\underset{\sim}{n}$$

$\square$

$$\chi_n(g) = \frac{e^{i(n+\frac{1}{2})r} - e^{-i(n+\frac{1}{2})r}}{e^{ir/2} - e^{-ir/2}}$$

$$\chi_n(g) = e^{r\underset{\sim}{n}\cdot\underset{\sim}{J}}$$

$$= f(r)$$

Fourier transform is <u>awful</u>

Fix it up with "twist"

Motivation:

Note locally

$\sqrt{\cdot}$ $\quad$ $\sqrt{\cdot}$ $\quad$ $\shortparallel\; j(X)$

$$\int_{SO_3} f(e^X)\, g(e^X)\, \left|\det (d\exp|_X)\right| dX$$

characters

$$= \int_{SO_3} f(y)\, g(y)\, d\mu(y)$$

by change of variables

$$y = \exp(X).$$

Fact

$$j(X) = \left(\frac{e^{ir/2} - e^{-ir/2}}{ir} \right)^2$$

$$X = r\,\underset{\sim}{n}\cdot\underset{\sim}{J}$$

$$\Rightarrow \quad \chi_n(e^X)\sqrt{j(X)} = \frac{e^{i(n+\frac{1}{2})r} - e^{-i(n+\frac{1}{2})r}}{ir}$$

Looks like the Fourier transf.

of a surface measure μ

(this will be corresponding orbits of the coadjoint action)

<u>Def</u> If μ is a surface measure (a measure on a 2d mfld.?) on S^2_R then its <u>Fourier</u>

transform is

$$\hat{\mu}(\underset{\sim}{k}) := \int_{S_R^2} e^{i\langle \underset{\sim}{k}, \underset{\sim}{x} \rangle} \, d\mu(\underset{\sim}{x})$$

Lemma μ_R standard measure on S_R^2 is: $\hat{\mu}_R(\underset{\sim}{k}) = 2\pi R \left(\frac{e^{iR\lambda} - e^{-iR\lambda}}{i\lambda} \right.$

where $\lambda = |\underset{\sim}{k}|$.

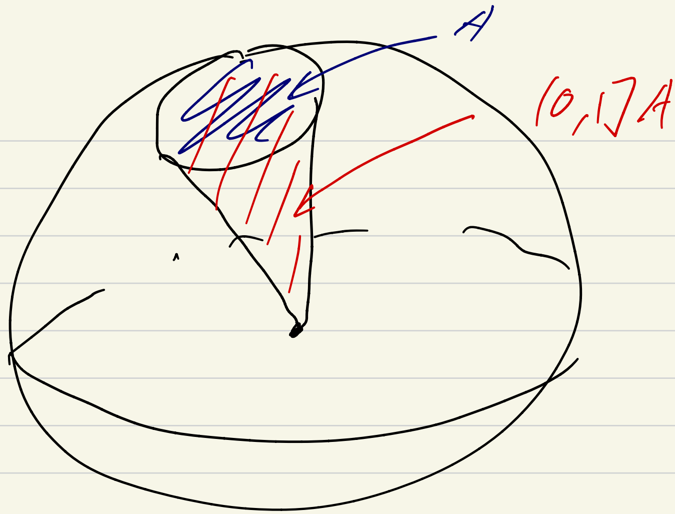
$A \in \mathcal{B}(S_R^2)$ then

$$\mu_R(A) := n\mu\left([0,1] A \right)$$

$n = 3, \; S_R^{n-1}$

Lebesgue measure

$$\int_0^1 \iint_A r^2 \sin\theta \; d\theta \; d\phi \; dr$$



Pf Observe that μ_R is radial so

$$g \in SO_3, \quad \mu_R(gA) = \mu_R(A)$$

so $g_*(\mu) = \mu \quad \forall g \in SO_3.$

Claim: $\hat{\mu}_R(g\underset{\sim}{k}) = \hat{\mu}_R(\underset{\sim}{k})$

Pf: See Joan's comment.

So it suffices to compute

$$\hat{\mu}_R((0,0,\lambda)) = \int_{S_R^2} e^{i\langle(0,0,\lambda),\, \underset{\sim}{x}\rangle} \, d\mu_R(\underset{\sim}{x})$$

$i\lambda z$

$$z = \mathrm{proj}_z(\underset{\sim}{x})$$

$$= \int_{[-R,R]} e^{i\lambda z} \, d(\mathrm{proj}_z)_* \mu_R(z)$$

$$(\mathrm{proj}_z)_* \mu = 2\pi R \, dz$$

$$= 2\pi R \int_{-R}^{R} e^{i\lambda z} \, dz$$

$$= 2\pi R \, \frac{e^{+iR\lambda} - e^{-iR\lambda}}{i\lambda}$$

$\square$

So we see

$$2\pi R \cdot \chi_n(e^X)\sqrt{j(X)}$$

$$= \hat{\mu}_R(\underset{\sim}{k})$$

where $X = 2\underset{\sim}{n} \cdot \underset{\sim}{J}$ and

$R = n + \frac{1}{2}$, $n \in \mathbb{N}$.

$$\boxed{\begin{array}{l} \underline{\text{Main result:}} \\ \tilde{\chi}_n(e^{\underset{\sim}{k} \cdot \underset{\sim}{J}}) \\ \parallel \\ \chi_n(e^{\underset{\sim}{k} \cdot \underset{\sim}{J}})\sqrt{j(\underset{\sim}{k} \cdot \underset{\sim}{J})} \\ = \dfrac{\widehat{\mu_R}}{2\pi R}(\underset{\sim}{k}) \end{array}}$$

$$= \int_{S_R^2} e^{i\langle \underset{\sim}{k}, \underset{\sim}{x}\rangle} \frac{d\mu_R}{2\pi R}(\underset{\sim}{x})$$