Talk 3 – §2 of Venkatesh's notes

Recall: V_N irrep of $SO_3(\mathbb{R})$ of dim $2N+1$
χ_N char of V_N

Then $$(\chi_N \sqrt{j})(e^X) = \frac{e^{i(N+\frac{1}{2})s} - e^{-i(N+\frac{1}{2})s}}{is}$$

where $X \in \mathfrak{so}_3$, $|X| = s$ → wrt J_x, J_y, J_z basis

j Jacobian at X of $\exp : \mathfrak{so}_3 \to SO_3(\mathbb{R})$
↳ need to check....

- Denote $\hat{\mu} : \mathbb{R}^3 \to \mathbb{R}$ be the Fourier transform of the area measure of $S^2_R \subset \mathbb{R}^3$.

then $$\hat{\mu}(k) = \frac{e^{iR\lambda} - e^{-iR\lambda}}{i\lambda} \, 2\pi R$$

$\lambda = |k|$, $k \in \mathbb{R}^3$

$\Rightarrow$ If we normalise μ s.t. factor $2\pi R$ disappear and the total area is $2N+1 = \dim V_N$ then

$$(*) \quad \boxed{(\chi_N \sqrt{j})(e^X) = \int_{S^2_{N+\frac{1}{2}}} e^{i\langle \xi, X\rangle} \, d\xi}$$

→ Q: Where does $S^2_{N+\frac{1}{2}}$ come from?

Today: Ignore jacobian j, focus on explaining $S^2_{N+\frac{1}{2}}$ and μ ...

Corollary: $X = 0$

$$2N+1 = \dim V_N \overset{X \to 0}{=} (\chi_N \sqrt{j})(e^{0}) = \int_{S^2_{N+\frac{1}{2}}} d\xi = \text{area } S^2_{N+\frac{1}{2}}$$

Speculations of (*):

① From corollary, there exists $\{v_1, \ldots, v_N\}$ basis of V_N in bijection with partition

$$S^2_{N+\frac{1}{2}} = \bigsqcup_{k=1}^{2N+1} \mathcal{O}_k$$

s.t. $\mu(\mathcal{O}_k) = 1$.

② If each v_k satisfies

$SO_3 \curvearrowright V_N \overset{\text{diff}}{\rightsquigarrow} \mathfrak{so}_3 \curvearrowright V_N$ $\quad X \cdot v_k \simeq i\langle \xi_k, X \rangle v_k$ for some $\xi_k \in \mathcal{O}_k$,

then $e^{X} \cdot v_k \simeq e^{i\langle \xi_k, X\rangle} v_k$.

$$e^{\begin{array}{c} \begin{array}{ccc} v_1 & \cdots & v_k \end{array} \\ \begin{bmatrix} i\langle \xi_1, X\rangle & & \\ & \ddots & \\ & & \end{bmatrix} \end{array}} = \begin{bmatrix} e^{i\langle \xi_1, X\rangle} & & \\ & \ddots & \\ & & \end{bmatrix}$$

then $\mathrm{Tr}(e^X) = \sum e^{i\langle \xi_k, X\rangle} = \int_{S^2_{N+\frac{1}{2}}} e^{\langle \xi, X\rangle} d\xi$

$\Longrightarrow$ Decomposition $S^2_{N+\frac{1}{2}} = \bigsqcup \mathcal{O}_k$

correspond to diagonalisation of $SO_3(\mathbb{R})$-action

- But the above picture is not correct

as V_N is then sum of 1-dim rep

- Better way to speculate: V_k can be decomposed as sum of X-eigenvectors, each with eigvalue $i\langle \xi, X\rangle$ for some $\xi \in \mathcal{O}_k$.

Why would this speculation fix this issue?

$$[X, Y] \cdot v_k = i\langle \xi_{1(1)}, [X, Y]\rangle v_k \neq 0$$

$$[X, Y] v_k = XYv_k - YXv_k = 0$$

$$i_X : \mathcal{O}_k \to \mathbb{R} \quad \xi \mapsto \langle \xi, X\rangle$$

$$T \qquad \ell_X$$

$$v_k = \sum_{i=1}^{r} \alpha_i t_i \qquad X \cdot t_i = i\langle \xi_i, X\rangle t_i$$

General formula :

- G Lie grp, $\xi \in \mathfrak{g}^*$ $\quad \mathrm{ad} \cdot \mathfrak{g} \to$ —— $[,]$

Adjoint action $\mathrm{Ad} : G \to GL(\mathfrak{g})$

- Co-adjoint action $\mathrm{Ad}^* : G \to GL(\mathfrak{g}^*)$

by $\langle \mathrm{Ad}(g)v, f\rangle = \langle v, \mathrm{Ad}^*(g^{-1})f\rangle$

- Let $\mathcal{O}$ be orbit of $\xi \in \mathfrak{g}^*$. $\quad$ differentiate

$\hookrightarrow$ manifold (?)

$$\mathrm{ad}^* : \mathfrak{g} \to GL(\mathfrak{g}^*)$$

$$\gamma \qquad X \in \mathfrak{g} \qquad v \in \mathfrak{g}^*$$

$$\gamma'(0) = X,\ \gamma(0) = 1 \qquad \longleftarrow \exp(tX) : I \to G$$

$$\mathrm{ad}^*(X)\, v := \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}^*(\exp(tX))\, v$$

- Claim : $\mathrm{Tan}_\xi \mathcal{O} = \{\mathrm{ad}^*_X(\xi) : X \in \mathfrak{g}\}$.

Proof : path $\gamma_X(t) = \mathrm{Ad}^*_{\exp(tX)}(\xi) \qquad X \in \mathfrak{g}$

$\hookrightarrow$ exp of X

$\gamma_X(0) = Ad^*_1(\xi) = \xi$.

$\gamma_X'(0) = ad^*_X(\xi) \in T_\xi(\mathcal{O})$.

Thus, we have defined a linear map

$$\mathfrak{g} \longrightarrow Tan_\xi(\mathcal{O})$$
$$X \longmapsto \gamma_X'(0)$$

Its kernel a those $\underbrace{X \in \mathfrak{g} \text{ s.t. } ad^*_X(\xi) = 0}$.

$ad^*_{[X,Y]} = [ad^*_X, ad^*_Y]$ $\qquad \mathfrak{g}_\xi$: Lie subalg of $\mathfrak{g}$.

- $\mathfrak{g}_\xi$ is Lie alg of G_ξ, stabiliser of ξ under Ad^*.

Indeed, $\exp(\mathfrak{g}_\xi) \subset G_\xi \rightsquigarrow T_{id} G_\xi \supset \mathfrak{g}_\xi$.

Conversely, if $\gamma : I \to G_\xi \rightsquigarrow \gamma(t) \in G_\xi$

i.e. $Ad^*_{\gamma(t)} \xi = \xi \ \forall t$.

diff $\left.\frac{d}{dt}\right|_{t=0}$ $\hookrightarrow ad^*_{\gamma'(0)} \xi = 0 \Rightarrow \gamma'(0) \in \mathfrak{g}_\xi$. ⊣

- Injection $\mathfrak{g}/\mathfrak{g}_\xi \hookrightarrow Tan_\xi(\mathcal{O})$

$\mathcal{O} = G/G_\xi$ so $\mathfrak{g}/\mathfrak{g}_\xi$ and $Tan_\xi \mathcal{O}$ have same dim. $\Rightarrow$ surjective ▨

$\mathcal{O} = G \cdot \xi$

— There is a natural G-invariant nondeg symplectic form ω on $\mathcal{O}$ (i.e. an alternating non-deg bilinear form on $T_u\mathcal{O}$) defined by

For each $t_1, t_2 \in \mathrm{Tan}_\xi \mathcal{O}$, choose $X, Y \in \mathfrak{g}$ s.t. $X \cdot \xi = t_1$, $Y \cdot \xi = t_2$.

$\mathrm{ad}^*(Y)\xi$

$$\omega(t_1, t_2) := \xi([X, Y]).$$

well-defined, does not depend on the choice of X and Y.

canonical, i.e. does not depend on ξ
replace ξ by anything in $\mathcal{O}$.

— ω is G-invariant: $g \in G$, on $T_{g\cdot\xi}(\mathcal{O})$ we have

$$\omega_{g\cdot\xi}(g_* t_1, g_* t_2) = \omega_\xi(t_1, t_2)$$

where $g_* : \mathrm{Tan}_\xi(\mathcal{O}) \longrightarrow \mathrm{Tan}_{g\xi}(\mathcal{O})$ because $g : \mathcal{O} \to \mathcal{O}$

ad^* $\qquad$ Ad^*

$\Rightarrow (\mathcal{O}, \omega)$ is a symplectic manifold.

• Example: - Adjoint action of SO_3 on so_3 is iso to the rotation action of SO_3 on $\mathbb{R}^3$

$\Rightarrow$ Co-adjoint action (conjugate transpose of adjoint action) is also rotation on $\mathbb{R}^3$

$\Rightarrow$ Co-adjoint orbits correspond to spheres in $\mathbb{R}^3$

$\mathcal{O}$ ↑ always even dim. dim 2

Theorem (Kirillov, others) G connected Lie grp (either nilpotent or semisimple). π tempered rep of G (i.e. π "lives" in $L^2(G) \circlearrowleft G$). Then there is an orbit $\mathcal{O}$ of G on $\mathfrak{g}^*$ s.t. $\left(\chi_\pi \sqrt{j}\right)(e^X) =$ Fourier trans of $\left(\frac{\omega}{2\pi}\right)^d$ on $\mathcal{O}$

ω symplectic form to $\mathcal{O}$

$2d =$ real dim of $\mathcal{O}$.

Upshot: { Know where $\mathcal{O}$ comes from
{ —— how to define the measure

Describe — explicitly for SO_3:

(exer 2.3.7, p. 8) —

next time

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