Pseudo-diff. operators

Recall Kirillov char. formula: (KXF)

$$\left(\chi_{V_N} \cdot \sqrt{j}\right)\left(e^X\right) = \int_{\mathcal{O}} e^{i\langle \xi, X\rangle} d\xi$$

where $\mathcal{O}$ a sphere in $\mathfrak{so}_3^*$
with area $\dim V_N$.

Today look at more generalised setting where

$$V_N \rightsquigarrow L^2(\mathbb{R})$$

and

$$\mathcal{O} \rightsquigarrow \mathbb{R}^2$$

These will be connected by pseudo-diff. op's

Later section we'll see that
KXF will come from a formula

for the trace of these operators.

(*) Fourier transf.

Our goal Write $f \in L^2(\mathbb{R})$ as a sum of "localised functions" $f_i \in L^2(\mathbb{R})$ for which both f_i and $\hat{f}_i$ are localised (supported in some small interval).

Why? The f_i will be eigenfunctions of a diff. operator.

e.g.
$$\widehat{\left(\frac{d}{dx} f\right)}(\xi) = -i\xi \cdot \hat{f}(\xi).$$

More generally cst coeff. diff. op. acts on Fourier transf. by multiplication by a polynomial.

For non-cst diff. op. this won't work but if f is localised we can get it to work.

<u>Problem</u> <u>Lemma</u> g and $\hat{g}$ can't both be compactly supported.

E.g. $g(x) = \chi_{[-N/2, N/2]}$

$$\hat{\chi}_{[-N/2, N/2]}(\xi) = \frac{2 \cos(N/2\, \xi)}{i\xi}$$

More precisely: $I, J \subset \mathbb{R}$ intervals with lengths L and M.

$$\operatorname{supp} \chi_I f \subset I$$

$$\operatorname{supp} \chi_J \hat{f} \subset J$$

Also, $$\left(\chi_J \hat{f}\right)^{\vee} = \check{\chi}_J * f$$

Two operations

(1) $f \overset{\phi}{\mapsto} \chi_I f$, localising f

(2) $f \overset{\psi}{\mapsto} \check{\chi}_J * f$, localising $\hat{f}$

Problem (1) and (2) don't commute so

$$\psi \circ \phi(f) \neq \phi \circ \psi(f)$$

A way out is using pseudo-diff. op's.

(*) Pseudo-diff op's

$a, b \in C^\infty(\mathbb{R})$ approximate

χ_I and χ_J

Calculate

"take back to f"

$$\big(\underbrace{b(\xi)\,\hat{f}}\big)^{\vee}(x)$$

localise $\hat{f}$

$$= \int b(\xi)\,\hat{f}(\xi)\,e^{-ix\xi}\,d\xi$$

localise the result:

$$a(x)\,\big(b(\xi)\,\hat{f}\big)^{\vee}(x)$$

$$= \int \underbrace{\overset{\simeq\, \chi_I}{a(x)}\;\overset{\simeq\, \chi_J}{b(\xi)}}_{a(x,\xi)}\,\hat{f}(\xi)\,e^{-ix\xi}\,d\xi$$

<u>Def</u> $a(x,\xi) \in C_c^\infty(\mathbb{R}^2)$

$$(\mathrm{Op}(a)f)(x) = \int a(x,\xi)\,\hat{f}(\xi)\,e^{-ix\xi}\,d\xi$$

called a pseudo-diff. op. with symbol a.

Want $a(x,\xi)$ to be a smooth

approx. of $\chi_{I\times J}$ so

that $Op(a)$ "localise" $Op(a)f$

to I and $\widehat{Op(a)f}$ to J.

$f \mapsto \chi_I f$ is a proj. op.

so expect $Op(\chi_{I\times J})$

and so $Op(a)$ to be

so well:

$$Op(a)^2 \approx Op(a).$$

Q: Why do these imply $\widehat{Op(a)f}$ is localised?

Compute

(See Fact 3.3.2)

$$(Op(a)\,Op(b)\,f)(x)$$

$$= \int c(x,v)\,\hat{f}(v)\,e^{-ixv}\,dv$$

$$= (\mathrm{Op}(c)\, f)(x)$$

where $a(x,\xi)$, $b(y,v)$

are "regular functs"

and

$$c(x,v) = \iint a(x, v+s)\, e^{ist}\, b(x+t, v)\, ds\, dt$$

By Taylor exp and integral formula

$$= \sum_{l \geq 0} \frac{i^l}{l!} \partial_2^{\,l} a(x,v)\, \partial_1^{\,l} b(x,v)$$

$$= a(x,v)\, b(x,v) + i\, \partial_v a\, \partial_x b$$

$$(\Delta) \qquad - \frac{1}{2} \partial_v^2 a\, \partial_x^2 b + \text{h.o.t}$$

Degree of commutativity:

$$\underbrace{\mathrm{Op}(c)}_{=:\, a \cdot b} = \mathrm{Op}(a)\, \mathrm{Op}(b)$$

$$a \cdot b - b \cdot a = i(\partial_v a \, \partial_x b - \partial_x a \, \partial_v b)$$

$$+ \text{ h.o.t.}$$

Poisson bracket of a and b (connection to symplectic geom.)

<u>Projection op.</u>

$$I = [0, L], \quad J = [0, M]$$

ϕ a C^∞ approx. of

$$\chi_{[0,1]}$$

e.g.

$$\chi_{[0,1]} * \mathcal{J}_n$$

$$\mathcal{J}_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2 x^2} \to \delta(x)$$

$$a(x, v) \underset{\approx}{=} \chi_{I \times J} \phi\left(\frac{x}{L}\right) \phi\left(\frac{v}{M}\right)$$

Then from (Δ):

$$a \cdot a = a^2 + \frac{1}{LM} \cdot (\text{der. of } \phi)$$

$$+ \frac{1}{L^2 M^2} (2^{nd} \text{ der. of } \phi)$$

$$+ \text{h.o.t.}$$

so if $LM \gg 1$,

$$a \cdot a \approx a^2$$

so $\quad \mathrm{Op}(a)^2 \approx \mathrm{Op}(a^2)$

$$\approx \mathrm{Op}(a)$$

since a is an approx. of a char. function.

U: Answer to $\overbrace{Op(a) f}$ being locally supported.

$$\text{supp } \mathcal{F}\left(\chi_I \cdot \mathcal{F}^{-1}\left(\chi_J \cdot \mathcal{F}(f)\right)\right) \overset{?}{\subset} J$$

$$\left| \left(\underbrace{\mathcal{F}(\chi_I)}_{g} * \left(\underbrace{\chi_J \cdot \mathcal{F}(f)}_{h}\right) \right)(\xi) \right|$$

$$= \left| \int g(\xi - t)\, h(t)\, dt \right|$$

$$= \left| \int_J \hat{\chi}_I(\xi - t)\, \hat{f}(t)\, dt \right|$$

$$\left| \hat{\chi}_{[-N/2,\, N/2]}(\xi) \right| = \left| \frac{2 \cos(N/2\; \xi)}{i \xi} \right| \lesssim \frac{2}{|\xi|}$$

$$\leq \int_J \frac{2}{|\xi - t|} |g(t)| \, dt$$

$$\leq M \int_J \frac{1}{|\xi - t|} \, dt$$

≈ 0 for $|\xi - t| \gg 0$

or ξ far away from J.