The Heisenberg group and its unitary

Recall We defined $Op(a)$ on $L^2(\mathbb{R})$ which were localisation operators of f and $\hat{f}$.

Speculation: Decomposing $L^2(\mathbb{R})$ into localised functions which would correspond to eigenfunctions of some group action

we get an idea of what this group action is.

Saw that $\hat{f}$ localised meant

$$f \simeq e^{i\xi x} \quad \text{as}$$

$$F^{-1}(e^{i\xi x}) = \delta(x-\xi).$$

Which is an eigenvector of translation $(\tau_y \cdot f)(t) = f(t+y)$

and

$$(m_y \cdot \delta(x-\xi))(x) = e^{ixy}\,\delta(x-\xi) = e^{i\xi y}\,\delta(x-\xi)$$

The group generated τ_z, m_y along with scalar multiplication is isomorphic to the Heisenberg group

$$\mathrm{Heis} := \left\{ \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{R} \right\}$$

generated by

$$U_x = \begin{pmatrix} 1 & x & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix},$$

$$V_y = \begin{pmatrix} 1 & 0 & 0 \\ & 1 & y \\ & & 1 \end{pmatrix},$$

and $$W_z = \begin{pmatrix} 1 & & z \\ & 1 & \\ & & 1 \end{pmatrix}$$

lies in $Z(\mathrm{Heis})$.

Has relations:

$$U_x V_y = V_y U_x W_{xy}$$

2-step nilpotent:

$$[[A,B],C] = 1$$

$$[A,B] := ABA^{-1}B^{-1}.$$

Lie algebra:

$$\mathrm{Lie}(\mathrm{Heis}) = \left\{ \begin{pmatrix} 0 & x & y \\ & 0 & z \\ & & 0 \end{pmatrix} \;\middle|\; x,y,z \in \mathbb{R} \right\}$$

Fact:
$$\exp \begin{pmatrix} 0 & x & y \\ & 0 & z \\ & & 0 \end{pmatrix} = \begin{pmatrix} 1 & x & z + \frac{xy}{2} \\ & 1 & y \\ & & 1 \end{pmatrix}$$

giving a more symmetric coordinatization

Action on L^2

The action giving the previous action on L^2 is:

$$U_x \mapsto \tau_x$$

$$V_y \mapsto m_y$$

$W_z \mapsto$ scalar multipli. by e^{iz}.

$\rightsquigarrow$ to unitary representation:

$$\mathrm{Heis} \longrightarrow U(L^2(\mathbb{R}))$$

<u>Note</u>: Denote by H a separable Hilbert space.

<u>Def</u> A <u>unitary rep.</u> of G on H is a homomorphism

$$G \to \mathrm{Unitary}(H).$$

s.t. action

$$G \times H \to H$$

is cts.

U is a bounded lin. operator s.t. $U = U^*$.

Def A unitary rep H of G is <u>irreducible</u> if H has no <u>closed</u> G-invariant subspaces.

Def Denote by $\hat{G}$ the set of unitary irreps of G up to isomorphism.

Remark $\hat{G}$ can be quite pathological in some cases.

E.g. $G = \mathrm{Heis}_{\mathbb{Z}}$, $x, y, z \in \mathbb{Z}$.

Take a similar action as Heis on $L^2(\mathbb{R})$ but instead on $L^2(\mathbb{Z})$:

Parametrised by $\alpha, \beta \in \mathbb{R}$ and denoted $V(\alpha, \beta)$.

$$(U_x f)(t) = f(t+x)$$

$$(V_y f)(t) = e^{2\pi i (\alpha t + \beta) y} f(t)$$

$$(W_z f)(t) = e^{2\pi i \alpha z} f(t)$$

where $x, y, z \in \mathbb{Z}$.

Not hard to see that

for $\beta' = \beta + n\alpha + m, \quad n, m \in \mathbb{Z}$

that $V(\alpha, \beta') \cong V(\alpha, \beta)$.

$\Rightarrow$ Fixing α, $\beta \in \mathbb{R} / \langle \mathbb{Z} + \alpha \mathbb{Z} \rangle$

is a dense subset of $\mathbb{R}$

so the indexing set (α, β)

is quite awful!

Ex: $V(\alpha, \beta)$ is irreducible if α is irrational.

To prove this we need

Lemma (Schur): A unitary rep H of G is irreducible $\Leftrightarrow$ any bounded operator $A \in \mathrm{End}(H)$ commuting with the action of G is scalar.

Pf ($\Leftarrow$) By contrad., assume the rep is not irreducible so $\exists H_1$, a closed G-inv. subspace.

Since H is unitary we get the orthogonal decomposition:

$$H = H_1 \oplus H_1^{\perp}$$

So $A = \mathrm{proj}_{H_1}$

will commute with $\{v \in H \mid \langle u, v\rangle = 0, u \in H_1\}$

the action of G but isn't scalar. ↯

WLOG $v \in H_1$ or $\in H_1^\perp$

$gv \in H_1$	$v \in H_1^\perp$
$\mathrm{proj}_{H_1}(gv) = gv$	$\mathrm{proj}_{H_1}(gv) = 0$
$g\,\mathrm{proj}_{H_1}(v) = gv$	$g\,\mathrm{proj}_{H_1}(v) = g0$
	$= 0$

($\Leftarrow$) Idea in finite dim case was to find "eigenspaces" V of A.

$AV \subset V$

First note that:

$$gA = Ag \Rightarrow A^* g^* = g^* A^*$$

as $g^* = g^{-1}$

this means A^* also commutes with the action ($G^{-1} = G$).

Observe that

$$A = \frac{(A + A^*)}{2} + i\left(\frac{A - A^*}{2i}\right)$$

where

self adj.

so first look at case where A is self-adjoint.

"<u>Spectral theorem</u>" : A is a self-adj. op. on H then for every Borel set $T \subset \mathbb{R}$, it makes sense to talk of H_T = " sum of eigenspaces with eigenvalues in T and

read spectral values

the spectrum of A on H_T is contained in $\overline{T}$.

λ s.t. $(A - \lambda I)$ is invertible.

$$p_n \to \chi_T$$

$$p_n(A)\, H \longrightarrow H_T$$

$$Av = \lambda v \Rightarrow p_n(A)v \to \chi_T(\lambda)v$$
$$\lambda \in T \qquad = \begin{cases} 0, & \lambda \notin T \\ v, & \lambda \in T \end{cases}$$

$$\boxed{p_n(A)v = p_n(\lambda)v \to \chi_T(\lambda)v}$$

Assuming H is irreducible,

$$H_T = 0,\ H$$

as H_T is a closed G-inv. subspace (as $Ag = gA$, $\forall g$).

Now take a sufficiently large

interval T_0 s.t. $H_T = H$

(so $\sigma(A) \subset \bar{T}_0$).

Shrink the size of T_0, take

interval $T_1 \subset T_0$, $|T_1| = |T_0|/2$

so $T_0 = T_1 \cup T_1'$

$$H_{T_0} = H_{T_1} \oplus H_{T_1'} = H$$

WLOG let $H_{T_1} = H$.

Then repeat with T_1

$$\Rightarrow T_2, T_2'$$

$\vdots$

So eventually we get

$$\sigma(A) \subset T_0 \cap T_1 \cap T_2 \cap \cdots$$

$$= \{\lambda\}$$

$|T_0| = l$

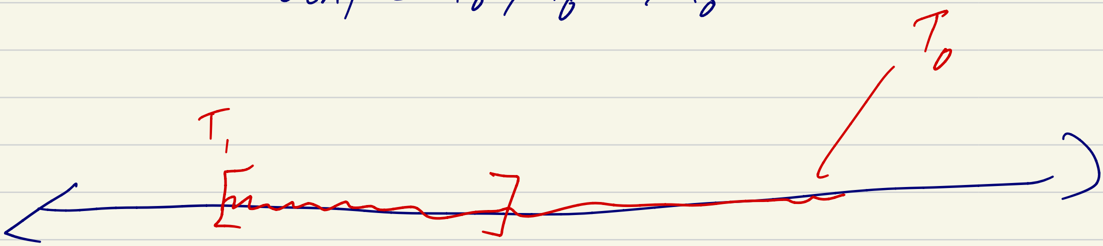
$|T_i| = l/2^i$

$$\Rightarrow \quad A = \lambda \, Id$$

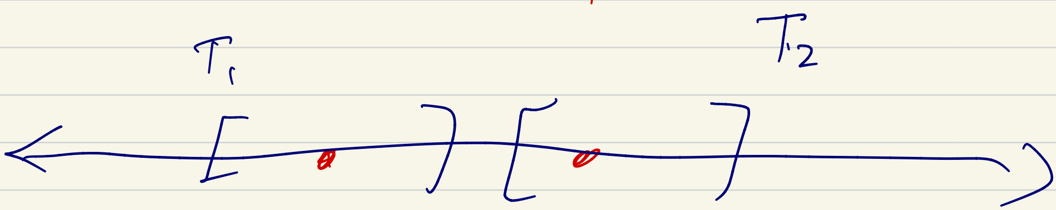
$\sigma(A) \neq \emptyset$ as A is self-adj.

$$\left(H_{T_0} = H_{T_0'} = H_{T_0^2} = \cdots = H \right)$$

so $\sigma(A) \subset T_0, T_0', T_0^2 \cdots$



suppose $H_{T_1} = 0$



$$H_{T_1} = H, 0$$

so $\sigma(A) \subset T_1$ or $\sigma(A) \subset T_1^c$

If A wasn't self-adj.

then using

$$A = \underbrace{\left(A + A^*\right)/2}_{\text{self-adj}} + i\underbrace{\left(\frac{A - A^*}{2i}\right)}_{\text{self-adj}}$$

$$\Rightarrow A = \lambda_1 \mathbb{1} + \lambda_2 \mathbb{1}$$

$$= (\lambda_1 + \lambda_2)\mathbb{1}$$

$\square$

<u>Ex.</u> $V(\alpha, \beta)$ is irreducible if α is irrational.

Pf Idea: V_y is an ∞-dim. diagonal matrix with different entries.

A, a bounded op. commuting with action of G.

$$(A V_y)(f)(t) = (V_y A)(f)(t)$$

$$\text{LHS} = A\left(e^{2\pi i(\alpha t+\beta)y} f(t)\right)(t)$$

$$\text{RHS} = e^{2\pi i(\alpha t+\beta)y} (Af)(t)$$

$$\Rightarrow \quad (Af)(t) =$$

$$e^{-2\pi i \alpha t y} A\left(e^{2\pi i \alpha t y} f(t)\right)(t)$$

My sketch (Ex 4.4.13)

An ON basis for $L^2(\mathbb{R})$:

$$\left\{ \psi_{n,k} = e^{2\pi i t n} \chi_{[k,k+1]} \right\}_{n,k=-\infty}^{\infty}$$

$$(V_y f)(t) = e^{2\pi i t y} f(t)$$

$p \in \mathbb{Z}$:

$$V_p \psi_{n,k} = \psi_{n+p,k}$$

$$A \psi_{n,k} := \sum_{s,t} a_{n,k}^{s,t} \psi_{s,t}$$

$$\Rightarrow \quad A V_p \psi_{m,l} = A \psi_{m+p,l}$$

$$= \sum_{s,t} a_{m+p,l}^{s,t} \psi_{s,t}$$

$$V_p A \Psi_{m,l} = V_p \sum_{r,u} a_{m,l}^{r,u} \Psi_{r,u}$$

$$= \sum_{r,u} a_{m,l}^{r,u} \Psi_{r+p,u}$$

Equating terms :

$$\boxed{s = r+p, \quad t = u}$$

$$a_{m+p,l}^{s,t} = a_{m,l}^{s-p,t}$$

$s = m+p$

$\forall\, s,t,m,l,p \in \mathbb{Z}$ $\quad \Longleftrightarrow$

$s-p = m$

$t, l = 0 \quad , \quad s, m = 0 \quad :$

$$a_{p,0}^{0,0} = a_{0,0}^{-p,0}$$

$p \in \mathbb{Z}$:

$$U_p \, \psi_{n,k} = e^{2\pi i n (x+p)} \chi_{[k,k+1]}$$

$$= e^{2\pi i n x}$$

$\boxed{\text{Next time:}}$ get character of representation.