<u>Last time</u>: – Define a unitary rep of the Heisenberg group on $L^2(\mathbb{R})$:

Heis contains subgroups $U_x = \begin{pmatrix} 1 & x & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$, $V_y = \begin{pmatrix} 1 & 0 & 0 \\ & 1 & y \\ & & 1 \end{pmatrix}$, $W_z = \begin{pmatrix} 1 & 0 & z \\ & 1 & 0 \\ & & 1 \end{pmatrix}$

and these act on $L^2(\mathbb{R})$ by

$U_x \mapsto \tau_x$ translation $(f(t) \mapsto f(t+x))$

$V_y \mapsto m_y$ multiplication $(f(t) \mapsto e^{iyt} f(t))$

$W_z \mapsto$ scalar multiplication by e^{iz}.

UNSOLVE: <u>This rep is irreducible</u>, using Schur's lemma.

- Part where we don't understand last time:

. G Lie group, H Hilbert space, we want to impose some continuity condition on the rep $G \to \mathrm{End}(H)$

. The point is that asking for $G \to \mathrm{End}(H)$ to be continuous is too strong, in the sense that the translation action $\mathbb{R} \to \mathrm{End}(L^2(\mathbb{R}))$ is not continuous

↳ can see this by noticing that

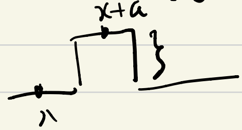
→ Topology given by operator norm

a very small translation by $r \in \mathbb{R}$ is very far away from the identity operator.

$a \mapsto T_a$ $\qquad \|T_a - id\|_{op} \geq \|(T_a - id)f\| / \|f\|$ is large for good enough f

f indicator func on some interval of size $<< a$

$$\|(t_a - id)f\| = \left(\int_{\mathbb{R}} |f(x+a) - f(x)|^2 dx\right)^{1/2}$$



- In the end, the continuity condition we want is $G \times H \to H$ being continuous.

Today: Want to give character formula for Heis $\curvearrowright L^2(\mathbb{R})$ that resembles Kirillov's char formula.

§4.5, 4.6, 4.7

First problem: W_Z acts as a scalar on $L^2(\mathbb{R})$ so cannot define trace of W_Z.

Deal with this by working with the group algebra of $G = $ Heis.

Group algebra: G locally compact group, $\pi : G \to U(H)$ irreducible rep. Then can define $L^1(G) \curvearrowright^{\pi} H$ by

$$A \in L^1(G) \text{ then } \pi(A)v = \int_G A(g)(g \cdot v)\, dg$$

fix Haar measure of G (→ dg)

- $\pi(A)$ is a bounded operator $H \to H$, $\|\pi(A)v\| \leq \|A\|_{L^1} \cdot \|v\|$.
- $C_c^\infty(G) \subset L^1(G)$ is called group algebra of G.

Example: $G = \mathbb{R} \curvearrowright L^2(\mathbb{R})$ by translation then $\check{A}(x) := A(-x)$

$$(\pi(A)f)(t) = \int_{\mathbb{R}} A(x) f(x+t)\, dx = (f * \check{A})(t)$$

- Define a subspace H^∞ of H contains "smooth vectors" and H^∞ dense in H. What is purpose of this def?

⊛ Compute the character from the group algebra action:

$A \in C_c^\infty(\text{Heis})$, $(\pi(A)f)(t) := \int A\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} f(t+x) e^{i(ty+z)} \, dx \, dy \, dz$

$$B(x,y) := \int A\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} e^{iz} \, dz$$

then

$$(\pi(A)f)(t) = \int B(x,y) f(x+t) e^{ity} \, dx \, dy$$

Claim: $\pi(A)f$ is essentially $Op(\hat{B})f$, up to a few signs, where $\hat{B}$ is Fourier transform on $\mathbb{R}^2$.

Sketch: $\hat{B}(u,t) := \int B(x,y) e^{i(ux+ty)} \, dx \, dy$

So $(\pi(A)f)(t) = \int \hat{B}(u,t) e^{-iux} f(t+x) \, dx \, du$

$$= \int \hat{B}(u,t) e^{iut} \hat{f}(-u) \, du$$

$\hat{f}(-u) = \int_{\mathbb{R}} f(t+x) e^{i(t+x)(-u)} \, dx$

Similar to, up to minus sign of u,

$\hat{B}(u,t) = \hat{B}(-u,t)$

$Op(\hat{B})f := \int \hat{B}(x,\xi) \hat{f}(\xi) e^{-ix\xi} \, d\xi.$

↓

$Tr(Op(a)) = \int a(x,\xi) \, dx \, d\xi$

So, $\mathrm{Tr}\,\pi(A) \sim \mathrm{Tr}\left(\mathrm{Op}(\hat{B})\right) = \int \hat{B}(u,v)\,du\,dv \overset{!}{=} B(0,0).$

$$= \int A(0,0,z)e^{iz}\,dz$$

Actually, with the correct normalisation of Fourier trans of B

$$\boxed{\mathrm{Tr}(\pi(A)) = (2\pi)\int A(0,0,z)e^{iz}\,dz}, \qquad (1)$$

— Now, we want a character formula of G :

See $\chi : G \to \mathbb{C}$ as a distribution $L^1(G) \to \mathbb{C}$

So χ as a distribution is defined as

$$A \in L^1(G) \mapsto \int_G \chi(g)A(g)\,dg$$

So we should interpret (1) as $\chi = (2\pi)\,e^{iz}\,\delta_{z\text{-axis}}$

<u>Remark</u>: This suggests that τ_x and m_y have trace 0 for $x, y \neq 0$. And we can show this for τ_x :

Choose a basis for $L^2(\mathbb{R})$ in which τ_x has entries supported strictly above the diagonal.

for fix x, $\varphi_{n,k}(t) = e^{2\pi i nt}\,\delta_{[xk,\,x(k+1))}$

is a basis

so τ_x strictly above diagonal.

View the character as distribution on $\mathfrak{g}$:

- Recall: For Heis,

$$\exp\begin{pmatrix} 0 & x & y \\ & 0 & z \\ & & 0 \end{pmatrix} = \begin{pmatrix} 1 & x & z+\frac{xy}{2} \\ & 1 & y \\ & & 1 \end{pmatrix}$$ is a diffeomorphism

with the Haar measure being preserved i.e. the Jacobian is 1.

$$dx\,dy\,dz \longmapsto dx\,dy\,dz$$

- If we pull back the character by exp, we get a distribution on $\mathrm{Lie}(\mathrm{Heis}) \doteq \mathfrak{g}$

$$C_c^\infty(\mathfrak{g}) \xrightarrow{\exp} C_c^\infty(G) \xrightarrow{\chi} \mathbb{C}$$

$$\psi \longmapsto \exp^{-1}\circ\psi \longmapsto (2\pi)\int_G (\exp^{-1}\circ\psi)(0,0,z)\, e^{iz}\, dz$$

$$\underset{\text{change of variable formula}}{=} (2\pi)\int_{\mathfrak{g}} \psi(0,0,z)\, e^{iz}\, dz$$

Take Fourier transform of the above to get

$$\boxed{\chi(e^X) = \frac{1}{2\pi}\int_{\alpha,\beta\in\mathbb{R}} e^{i(\alpha x+\beta y+z)}\, d\alpha\, d\beta}$$

Because $\int \psi(0,0,z)\, e^{iz}\, dz \sim \int \hat{\psi}(\alpha,\beta,1)\, d\alpha\, d\beta$ ↳ Fourier trans in $\mathbb{R}^3$

$$\int_z \left(\int_{\alpha,\beta,t} \hat{\psi}(\alpha,\beta,t)\, e^{-itz}\, d\alpha,\beta,t\right) e^{iz}\, dz$$

$$= \int_{\alpha,\beta}\left(\int_{x,y,z} \psi(x,y,z)\, e^{i(\alpha x+\beta y+z)}\right) d\alpha\, d\beta$$

↓

$$\int_t e^{it(1-z)}\,dz = \delta(1-t)$$

Dirac delta func

$$\doteq \int_{x,y,z} \Big(\underbrace{\int_{\alpha,\beta} e^{i(\alpha x+\beta y+z)}\, d\alpha\, d\beta}_{\mathcal{X}(e^X)} \Big) \psi(x,y,z)\, dx\, dy\, dz$$