Last time: Saw that for Lie (Heis),

$$(\Delta) \quad \chi(e^X) = \frac{1}{2\pi} \int_{\alpha,\beta \in \mathbb{R}} e^{i(\alpha x + \beta y + z)} \, d\alpha \, d\beta$$

Today: How does this relate to the coadjoint orbits of Heis?

Coordinates (x,y,z) on Heis:

$$(x,y,z) \longleftrightarrow \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix}$$

coordinates on Lie (Heis):

$$(u,v,w) \longleftrightarrow \begin{pmatrix} 0 & u & w \\ & 0 & v \\ & & 0 \end{pmatrix}$$

on Lie (Heis)*:

$$(\alpha, \beta, \gamma) : (u, v, w) \longmapsto \alpha u + \beta v + \gamma w$$

Recall that:

$\mathrm{Lie}(\mathrm{Heis})^*$

$$\langle (u, v, w), \mathrm{Ad}^*_{(x,y,z)^{-1}} f \rangle$$

Heis

$\in \mathrm{Lie}(\mathrm{Heis})$

$$:= \langle \mathrm{Ad}_{(x,y,z)} (u, v, w), f \rangle$$

$$\mathrm{Ad}_{(x,y,z)} (u, v, w) = \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \begin{pmatrix} & u & w \\ & & v \\ & & \end{pmatrix} \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} & u & w + xv - yu \\ & & v \\ & & \end{pmatrix}$$

Let $e_1 := (1,0,0)$, $e_2 := (0,1,0)$, $e_3 := (0,0,1)$ be a basis $\mathrm{Lie}(\mathrm{Heis})$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be the standard dual basis corresponding to it.

Hence :

$$\begin{cases} \mathrm{Ad}_{(x,y,z)} e_1 = 1 \cdot e_1 - y \cdot e_3 \\ \mathrm{Ad}_{(x,y,z)} e_2 = 1 \cdot e_2 + x \cdot e_3 \\ \mathrm{Ad}_{(x,y,z)} e_3 = e_3 \end{cases}$$

As for the dual :

$$\langle e_i , \mathrm{Ad}^{*}_{(x,y,z)^{-1}} \epsilon_j \rangle := \langle \mathrm{Ad}_{(x,y,z)} e_i , \epsilon_j \rangle$$

$$= \begin{cases} \delta^1_j - y\,\delta^3_j , & i = 1 \\ \delta^2_j + x\,\delta^3_j , & i = 2 \\ \delta^3_j , & i = 3 \end{cases}$$

$$\mathrm{ad}^*_{(x,y,z)^{-1}}\, \varepsilon_1 = \varepsilon_1$$

$$\mathrm{ad}^*_{(x,y,z)^{-1}}\, \varepsilon_2 = \varepsilon_2$$

$$\mathrm{ad}^*_{(x,y,z)^{-1}}\, \varepsilon_3 = -y\cdot\varepsilon_1 + x\cdot\varepsilon_2 + \varepsilon_3$$

<u>Orbits</u>: In terms of (α,β,γ),

$\gamma \neq 0$ then

$$\mathrm{ad}_{(x,y,z)^{-1}}(\alpha,\beta,\gamma) = (\alpha,\beta,0) + (-y\gamma, x\gamma, \gamma)$$

$$= (\alpha - y\gamma,\ \beta + x\gamma,\ \gamma)$$

$$\mathcal{O}_{(\alpha,\beta,\gamma),\ \gamma\neq 0} = \{\mathbb{R}^2 \times \{\gamma\}\}$$

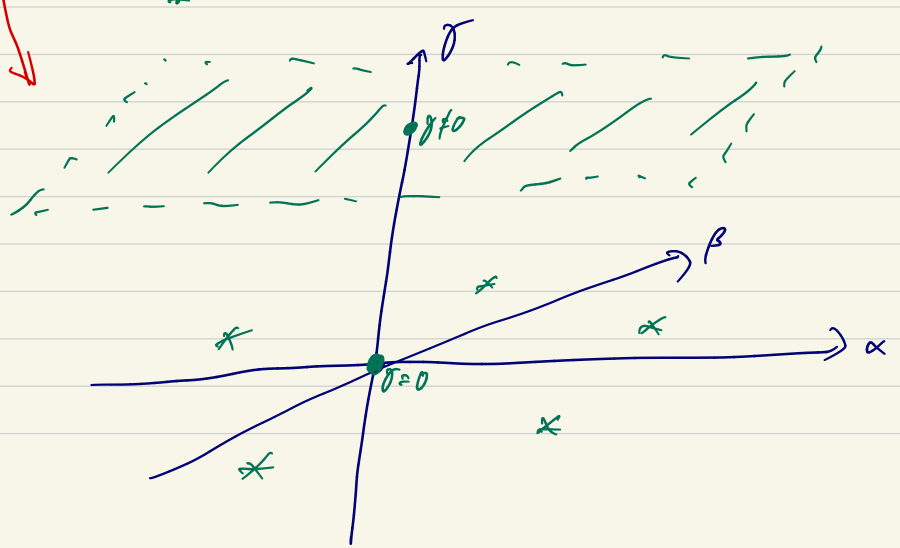
$$\mathcal{O}_{(\alpha,\beta,\gamma),\,\gamma=0} = \{(\alpha,\beta,0)\}$$

<u>$\chi(e^X)$</u> With this we can rewrite (Δ) for $\chi(e^X)$ as

$$\chi(e^X) = \frac{1}{2\pi}\int_{\alpha,\beta\in\mathbb{R}} e^{i(\alpha x+\beta y+z)}\, d\alpha\, d\beta$$

$$= \int_{\xi\in\mathcal{O}_{(\alpha,\beta,1)}} e^{i\langle \xi, X\rangle}\, \frac{d\alpha\, d\beta}{2\pi}.$$

$\mathcal{O}_1 \longrightarrow \mathcal{O}_{(\alpha,\beta,1)}$



Summary of section:

- Geometric decomposition of $L^2(\mathbb{R})$ corresponding to decomp. of $\mathbb{R}^2$ into rectangles.

- Find a gp such that it acts on each of these basis elements (the localised functions) as eigenvectors.

- Pseudo-diff. operators $Op(a)$ helped to decompose $L^2(\mathbb{R})$ into a basis through projectors.
- This was Heis.
- Then described character formula for this rep., one which looks like Kirillov formula.

- Saw how this connected to coadjoint orbits of Heis. These being $\mathbb{R}^2$ and pts.