Last time: • $\mathrm{Heis} \overset{\pi}{\curvearrowright} L^2(\mathbb{R})$
• Describe Kirrilov character formula for $\pi$.

Today: Kirrilov's theorem for Heis

**Theorem**: There is a 1-1 correspondence between $\widehat{\mathrm{Heis}}$ (space of irr unitary reps of Heis) and orbits $\Theta$ of $\mathrm{Heis} \curvearrowright \mathrm{Lie}(\mathrm{Heis})^*$

This correspondence is determined by: for each $\Theta$ there exists a unique rep $\pi_\Theta$ satisfying

$$\mathrm{Tr}\,\pi_\Theta(e^X) = \int_\Theta e^{i\langle \xi, X\rangle}\, d\xi \qquad (*)$$

Precisely, should understand this formula in the sense of distributions: - rep $\pi_\Theta$ of $H$ as a rep of $C_c^\infty(H)$.

i.e. $f \in C_c^\infty(H)$, $v \in (V, \pi_\Theta)$
$$f.v = \int_H f(h) hv\, dh$$
$$= \int_{\mathrm{Lie}(H)} f(X) e^X v\, dX$$

→ taking trace makes sense

LHS of (*)

$$\mathrm{Tr}(f) = \mathrm{Tr}\left(\int_{\mathrm{Lie}(H)} f(X) e^X\, dX\right)$$

- for RHS: character $\chi : H \to \mathbb{C}$ can be seen as a distribution $C_c^\infty(H) \to \mathbb{C}$ by

$$f \mapsto \int_H f(h)\chi(h)\, dh = \int_{\mathfrak{h}} f(X)\chi(X)\, dX$$

$$= \int_{X \in \mathfrak{h}} f(X) \left( \int_{\mathcal{O}} e^{i\langle \xi, X \rangle} d\xi \right) dX$$

$$= \int_{\mathcal{O}} \hat{f}(\xi)\, d\xi .$$

$\Rightarrow$ (*) should be understood as: $f \in C_c^\infty(\mathfrak{h})$

$$\mathrm{Tr}\left( \int f(X) e^X dX \right) = \int_{\mathcal{O}} \hat{f}(\xi)\, d\xi .$$

Remarks : – When $\mathcal{O}$ is a point, measure on $\mathcal{O}$ is point mass of measure 1.

– This theorem holds for any simply-connected, nilpotent Lie group, i.e. a connected Lie subgrp of $\begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$

(*) Construct $\pi_{\mathcal{O}}$ :

- For $\mathcal{O} = \mathcal{O}_{\alpha,\beta} = \{(\alpha, \beta, 0)\}$ then $\pi_{\mathcal{O}}$ is 1-dim.

$$\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \mapsto e^{i(\alpha x + \beta y)} .$$

- For $\mathcal{O} = \mathcal{O}_1 = \{(\alpha, \beta, 1) : \alpha, \beta \in \mathbb{R}\}$, we have seen that $L^2(\mathbb{R})$ works.

- For $\mathcal{O} = \mathcal{O}_\gamma$ where $\gamma \neq 1$ :
  - Notice that $\gamma \in \mathbb{R}^\times$ acts Heis by conjugating with $\begin{pmatrix} \gamma & & \\ & 1 & \\ & & 1 \end{pmatrix}$.

  $\rightsquigarrow$ A new rep of Heis : $\mathrm{Heis} \xrightarrow{\gamma} \mathrm{Heis} \xrightarrow{\mathcal{O}_1} U(L^2(\mathbb{R}))$

  $U_x \mapsto$ translation by $\gamma x$

$V_y \longmapsto$ multiplication by $y$

$W_z \longmapsto$ scalar multiplication by $e^{i\gamma z}$.

$$\begin{pmatrix}\gamma & & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & x & z \\ & 1 & y \\ & & 1\end{pmatrix}\begin{pmatrix}\gamma^{-1} & & \\ & 1 & \\ & & 1\end{pmatrix} = \begin{pmatrix}1 & x\gamma & z\gamma \\ & 1 & y \\ & & 1\end{pmatrix}.$$

So, we can repeat §4 to compute char formula for this rep $\rightsquigarrow$ corresponding co-adjoint orbit is $\mathcal{O}_\gamma$.

In other words, $\mathbb{R}^\times \curvearrowright \widehat{\text{Heis}}$ induces $\mathbb{R}^\times \curvearrowright$ orbits by $\gamma \cdot \mathcal{O}_\alpha = \mathcal{O}_{\gamma\alpha}$.

⊛ Classify all $\pi \in \widehat{\text{Heis}}$:

- By Schur's lemma: Action by the center $\begin{pmatrix}1 & & z \\ & 1 & \\ & & 1\end{pmatrix}$ induces operators on $\pi$ that commute with Heis

So by Schur's, we know $\begin{pmatrix}1 & & z \\ & 1 & \\ & & 1\end{pmatrix} v = \lambda_z v$ for all $v \in (V, \pi)$, for some $\lambda_z \in \mathbb{C}$.

We also have $\lambda_{z+z'} = \lambda_z \lambda_{z'}$. Because $\pi$ is continuous, so $\lambda : \mathbb{R} \to S^1$ is continuous and multiplicative.

cont $\text{Heis} \times V \to V$

$\rightsquigarrow \lambda(z) = e^{i\mu z}$ for some $\mu \in \mathbb{R}$.

- If $\mu = 0$, i.e. $\begin{pmatrix}1 & & z \\ & 1 & \\ & & 1\end{pmatrix}$ act as identity on $V$

i.e. rep factors through $\text{Heis}/Z(\text{Heis}) \simeq (\mathbb{R}^2, +)$, which is abelian, so all reps are 1-dim.

- If $\mu \neq 0$, WLOG $\mu = 1$ by $\mathbb{R}^\times \curvearrowright \text{Heis}$.

A theorem of Stone-von Neumann that the only irr reps with $\mu = 1$ is, up to iso, $L^2(\mathbb{R}) \circlearrowleft \text{Heis}$.

- If $(\pi, V)$ is an irrep of Heis then $\text{Heis} \xrightarrow{\gamma} \text{Heis} \xrightarrow{\pi} U(V)$ is also irrep because $\gamma$ is an automorphism on Heis.

<u>Remark</u>: $(\text{Heis}, L^2(\mathbb{R}))$ relates to quantum mechanics

- We get an action of $\text{Lie}(\text{Heis})$ on $L^2(\mathbb{R})^\infty$.
- $\text{Lie}(\text{Heis})$ generated by

$$U = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}.$$

Satisfying $[U, V] = W$

- Because $\exp(W)$ is central, $W$ acts by scalar

i.e. $[\underbrace{U}_{\text{position}}, \underbrace{V}_{\text{momentum}}] = \lambda \, \text{Id}$.

Now, ... Go to §7: Kirrilov's theorem for nilpotent grps.

Today: State the theorem.

- $G$ : connected, simply-connected, Lie grp with nilpotent Lie alg
  i.e. $\mathfrak{g} = \mathrm{Lie}(G)$ is nilpotent means
$$\mathfrak{g} \supset [\mathfrak{g},\mathfrak{g}] \supset [\mathfrak{g},[\mathfrak{g},\mathfrak{g}]] \supset \dots$$
  is eventually 0.

  - For such $G$, $\exp : \mathfrak{g} \to G$ is a diffeo.
    → can think of the grp structure being on $\mathfrak{g}$, given by the Campbell-Baker-Hausdorff formula:
$$\exp(X)\exp(Y) = \exp\left(X+Y+\tfrac{1}{2}[X,Y]+\dots\right)$$
  - $G$ is a closed subgrp of unipotent grp $\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}$

- Theorem:
$$\left\{\begin{matrix}\text{Irr unitary}\\ \text{reps of } G\end{matrix}\right\} \underset{\text{d}}{\overset{\sim}{\longleftrightarrow}} \left\{\begin{matrix}G\text{-orbits}\\ \text{on } \mathfrak{g}^*\end{matrix}\right\}$$

Kirilov char formula

$$\pi_{\mathcal{O}} \longleftarrow\!\!\!\mid \text{ orbit } \mathcal{O}$$

s.t. $X \in \mathfrak{g}$ $\quad \sqrt{j}\,(e^X)\, \mathrm{Tr}\, \pi_{\mathcal{O}}(e^X)$ (*)

$j$ Jacobian of $\exp : \mathfrak{g} \to G$ $\quad$ = Fourier Trans of measure talk 2 $\big($ Volume measure on $\mathcal{O}$ $\big)$.

$\left(\frac{\omega}{2\pi}\right)^d$ where $\omega$ symplectic form on $\mathcal{O}$, $2d = \dim \mathcal{O}$

- When $G$ is nilpotent: $\check{j} = 1$, and $(*)$ is understood in the sense of distributions: i.e., $f \in C_c^\infty(\mathfrak{g})$ then

$$\mathrm{Tr}\left(\int_X f(X)\,\pi(e^X)\,dX\right) = \int_{\mathcal{O}} \hat{f}(\xi)\,d\xi,$$

where $\hat{f}(\lambda) = \int f(X)\,e^{i\langle \lambda, X\rangle}\,dX$.

Note: the convergence of both sides is not obvious.