Today: Show the Kirillov thm for Nilpotent gps.

Due to nilpotence, the commutator is proper:

$$\mathfrak{g} \supsetneq [\mathfrak{g}, \mathfrak{g}]$$

giving a non-trivial Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathbb{R}$. Let $\mathfrak{g}_1 := \ker(\phi)$ then we get a subgroup $G_1 = \exp(\mathfrak{g}_1)$. Choosing $X \in \mathfrak{g} \setminus \mathfrak{g}_1$ we have

$$G = G_1 \rtimes \exp(\mathbb{R} \cdot X).$$

Proof:

$$1 \longrightarrow N = \exp(\mathfrak{g}_1) \xrightarrow{\iota} G \xrightarrow{\pi} Q = \exp(\mathbb{R}X) \to 1$$

$$\tau = \iota$$

a split SES

$\Rightarrow$ $\underline{G \cong N \rtimes Q}$

ι an injective map, π is surjective

$$\tau \circ \pi = Id_Q$$

($\tau = \iota$ the inclusion map).

Basis for $\mathfrak{g}$, $\beta = \{ X, \underbrace{a_1, \dots, a_\ell}_{\text{basis for } \ker(\phi) =: \mathfrak{g}_1} \}$

Define π as:

$$\pi(\exp(\lambda X + \sum \alpha_i a_i)) = \exp(\lambda X)$$

It is a gp hom. as

$$\exp(\lambda X + a) \exp(\sigma X + b)$$

$$\overset{BCH}{=} \exp((\lambda + \sigma)X + \underbrace{(a+b) + \text{comm}}_{\in \mathfrak{g}_1})$$

$$= \exp((\lambda + \sigma)X)$$

$$= \exp(\lambda X) \exp(\sigma X)$$

as $[\mathfrak{g}_1, \mathfrak{g}] \subseteq \mathfrak{g}_1$ and $a, b \in \mathfrak{g}_1$

__Note:__ $\ker(\pi) = \exp(\mathfrak{g}_1) = N$

$\square$

Idea: Proceed by induction. Take a rep π of G, restrict it to G_1 and then decompose it into irreducibles by using the theory of disintegration.

- Restriction remains irreducible.
- Splits up into a single orbit of an irred. σ of G_1 under the action of $\mathbb{R}$.

Obtain map $pr : \mathfrak{g}^* \to \mathfrak{g}_1^*$

$\mathbb{R} \curvearrowright G_1 \rightsquigarrow \mathbb{R} \curvearrowright$ space of irreps of $G_1 \ni \sigma$

(semidirect product)

$\sigma \xleftrightarrow[\text{Kirillov thm for } G_1]{} $ orbit $\mathcal{O} \subset \mathfrak{g}_1^*$

$$U\left(\exp(\mathbb{R}\cdot X) \curvearrowright pr^{-1}(\overline{\mathcal{O}}) \subset \mathfrak{g}^*\right)$$

w adj.

$=$ single orbit of $G \curvearrowright \mathfrak{g}^*$

$\rightsquigarrow$ Kirillov orbit thm π

Ex Heisenberg gp.

$$[\mathfrak{g}, \mathfrak{g}] = \{Z, Y-Z\}$$

To do §7.2 for

§7.3 Constructing the rep corresponding to an orbit ($\mathcal{O} \rightsquigarrow \pi_{\mathcal{O}}$)

Fix $\lambda \in \mathcal{O}$, note action $G \circlearrowright \lambda$:

$$g\lambda(gX) = \lambda(X) \qquad (*)$$

since

$$\langle X, \mathrm{Ad}_g^* \lambda \rangle := \langle \mathrm{Ad}_{g^{-1}} X, \lambda \rangle \iff g\lambda(X) = \lambda(g^{-1}X)$$

Let $g = e^{tY}$ then $(*)$:

$$\mathrm{Ad}^*_{e^{tY}} \lambda\left(\mathrm{Ad}_{e^{tY}} X\right) = \lambda(X)$$

$$\boxed{\begin{gathered}(d\,\mathrm{Ad})_1(Y) =: \mathrm{ad}_Y \\ (d\,\mathrm{Ad}^*)_1(Y) =: \mathrm{ad}^*_Y\end{gathered}}$$

taking the derivative at $t=0$:

$$(d\,\mathrm{Ad}^*)_1(Y)\lambda\left(\mathrm{Ad}_1 X\right) + \mathrm{Ad}_1^* \lambda\left((d\,\mathrm{Ad})_1(Y) X\right) = 0$$

$$\Rightarrow \quad \mathrm{ad}^*_Y \lambda(X) + \lambda(\mathrm{ad}_Y X) = 0$$

$$\Rightarrow \quad Y\lambda(X) + \lambda([Y, X]) = 0.$$

We see from this that

$$Y\lambda(X) = -\lambda([Y,X])$$

$\mathfrak{g}_\lambda := \mathrm{Lie}(\{g \in G \mid \mathrm{ad}^*_g \lambda = \lambda\})$

so $\mathrm{ad}^*_X \lambda = 0$, $g = e^{tX}$.

$\boxed{\text{cf. Talk 3}}$

$\mathrm{Lie}(\mathrm{Stab}_G(\lambda))$

so if $Y, X \in \mathfrak{g}_\lambda$,

$$\lambda([X,Y]) = 0$$

Meaning that $\lambda([\mathfrak{g}_\lambda, \mathfrak{g}_\lambda]) = 0$ so we have a Lie alg. homomorphism

$$\lambda : \mathfrak{g}_\lambda \to \mathbb{R}$$

exp. (both simply connected) $\rightsquigarrow$

$\boxed{\text{cf. Wikipedia - Lie alg. corresp.}}$

$$e^{i\lambda} : G_\lambda \to S^1$$

This is a <u>character</u> of $\mathrm{Stab}_G(\lambda) = G_\lambda$.

Note: $X, Y \mapsto \lambda([X,Y])$

$$\downarrow p$$

$\omega : \mathfrak{g}/\mathfrak{g}_\lambda \times \mathfrak{g}/\mathfrak{g}_\lambda \to \mathbb{R}$

which is alternating and non-degenerate. $\mathcal{O}$

$t_1, t_2 \in T_\lambda(\mathcal{O})$; $X, Y \in \mathfrak{g}$ s.t. $t_1 = X\cdot\lambda$, $t_2 = Y\cdot\lambda$ then

$$\omega(t_1, t_2) := \lambda([X,Y]).$$

So under the identification $\mathfrak{g}/\mathfrak{g}_\lambda \cong T_\lambda \mathcal{O}$ we get the canonical symplectic structure ω.

$$\begin{array}{cc} & q \\ q & \left(\begin{array}{c|c} 0 & -I \\ \hline I & 0 \end{array}\right) \end{array}$$

Induce the rep

Need to first extend the character as much as possible.

Want to find sub-algebra $q \supset \mathfrak{g}_\lambda$ s.t. $\lambda : q \to \mathbb{R}$ is a Lie alg. hom.,

i.e. $\lambda([q, q]) = 0$. So it has isotropic image in $\mathfrak{g}/\mathfrak{g}_\lambda$, i.e.

$\omega|_{p(q)} \equiv 0$ (where $(\mathfrak{g}/\mathfrak{g}_\lambda, \omega)$ is a symplectic structure).

$p : \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}_\lambda$

Def A polarization is a Lie subalg. $\mathfrak{g}_\lambda \subset \mathfrak{q} \subset \mathfrak{g}$ s.t. $\mathfrak{q}/\mathfrak{g}_\lambda$ is maximal isotropic (Lagrangian) for the form

$$X, Y \mapsto \lambda[X, Y].$$

Fact (Kirillov): Polarizations exist.

Let $\mathfrak{q}$ be a polarization, $Q := \exp(\mathfrak{q})$ and let

$$\chi : Q \to S^1$$

$$\chi(e^Y) := e^{i\lambda(Y)}, \quad Y \in \mathfrak{q}.$$

$$Q \overset{\chi}{\curvearrowright} \mathbb{C}$$

Then $\mathrm{Ind}_Q^G \chi$ is irreducible and gives $\pi_{\mathcal{O}}$. To verify this we compute the character.

Remaining Q's:

- What happens if you extend to the induced rep at G_λ for Heis instead of at Q?