Last time: – Defined pseudo-diff operators $Op(a)$ for some "differential operator" $a(x, \xi)$

where ξ viewed as "differential symbol", x as variable for the coef of the operator

$$(Op(a)f)(x) = \int a(x,\xi)\hat{f}(\xi)e^{-ix\xi}d\xi.$$

differentiate

e.g: $a = -i\xi$ $1..2n$

$$a\hat{f} = \widehat{\left(\frac{d}{dx}f\right)}(\xi) \Rightarrow Op(a)f = \frac{d}{dx}f$$

– Describe conditions for $Op(a)$ to approximate an operator that localise both $f \in L^2(\mathbb{R}^2)$ and $\hat{f}$.

(can only be approximation because f and $\hat{f}$ cannot be both localised) Last time $Op(a)f$ is localised but $\widehat{Op(a)f}$ is near 0 far away from some interval.

- a smooth approx of $\chi_{I \times J}$ where I, J closed intervals in $\mathbb{R}$
- Length $\ell(I)\ell(J) >> 1$ for $Op(a)$ to be a projection.

$\boxed{\text{Today}}$: – Discuss decomposition of $L^2(\mathbb{R})$ using pseudo diff operators.

§3.4, 3.5 Venkatesh notes – Make sense of what is a precisely

Now, $\mathbb{R}^2 = \bigsqcup_{i \in \mathbb{N}} \Theta_i$ where Θ_i = very large rectangle $I_i \times J_i$.

then $$\mathrm{Id}_{L^2(\mathbb{R})} = Op(1) = Op\left(\sum_{i \in \mathbb{N}} \chi_{\Theta_i}\right) \tag{1}$$

$$\simeq \sum_{i \in \mathbb{N}} \underbrace{Op(a_i)}_{\approx \text{ projection operator.}}$$ where a_i smooth approx of χ_{Θ_i}

- Let $V_i := \mathrm{Im}_{L^2(\mathbb{R})}\, Op(a_i)$ (viewed approximately as functions f supported on I_i and $\hat{f}$ supported on J_i)

Then from (1), $L^2(\mathbb{R}) = \sum_{i \in \mathbb{N}} V_i$

In fact, $L^2(\mathbb{R}) = \bigoplus_{i \in \mathbb{N}} V_i$ because $Op(a_i)$ are projection operators.

Indeed, since $Op(a_i) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ projection, we have
$$L^2(\mathbb{R}) = V_i \oplus \Big(1 - \underbrace{Op(a_i)}_{\sum_{j \neq i} Op(a_j)}\Big) L^2(\mathbb{R})$$

So if we have $f \in V_i \cap V_j$ then $f = 0 + f = f + 0$, a contradiction to above decomposition. ////

— Thus, we have a decomposition of $L^2(\mathbb{R})$ that is very geometric, i.e. it is based from partitioning $\mathbb{R}^2$ into rectangles!!

- In fact, the $\dim V_i$ is related to trace of $Op(a_i)$:

<u>Claim</u>: $Tr(Op(a_i)) = \int_{\mathcal{O}_i} a(x,\xi)\,dx\,d\xi \approx \text{Area}(\mathcal{O}_i)$.

<u>Proof</u>: Assume that f is Schwartz $\subset L^2(\mathbb{R})$:

$$(Op(a_i)f)(x) = \int a(x,\xi)\hat{f}(\xi)e^{-ix\xi}d\xi$$
$$= \int a(x,\xi)\left(\int e^{iy\xi}f(y)dy\right)e^{-ix\xi}d\xi$$
$$= \int \underbrace{\int a(x,\xi)e^{i(y-x)\xi}d\xi}_{K(x,y).\ \text{kernel}} f(y)dy$$

If $Op(a)$ is of trace class (i.e. $L^2(\mathbb{R})$ Hilbert $\to$ orthonormal basis $\{x_i\}$ $\to$ trace)

So trace of an integral kernel

is $Tr(Op(a)) = \int K(x,x)\,dx = \int a(x,\xi)\,dx\,d\xi.$

$\approx \text{Area}(\mathcal{O}_i)$

because $a_i \approx \chi_{\mathcal{O}_i}$ ▨

- What symbols a are allowed to make sense of
$$Op(a)Op(b) = Op(ab + i\partial_\xi a\,\partial_x b + \cdots)?$$

<u>Answer</u>: $a \in S_m$ = "diff operators of order $\le m$ with bounded coef".

$\hookrightarrow \sup_{x,\xi} \dfrac{|\partial_x^i \partial_\xi^j a(x,\xi)|}{(1+|\xi|)^{m-j}} \leq \text{const}(i,j) \quad \forall i,j \geq 0$

<u>Fact</u>: 1) If $a \in S_m$, $b \in S_{m'}$

then $\exists\, c \in S_{m+m'}$ so $Op(a)\,Op(b) = Op(c)$.

2) $c \simeq \sum_{N=0}^{\omega} \frac{i^N}{N!} \partial_\xi^N a \, \partial_x^N b$ for large ω.

<u>Remark</u>: The theory is not symmetric in the sense that it doesn't work if $\mathbb{R}^2$ is partitioned into rotated rectangles, for example.

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§4 of Venkatesh's lectures

<u>Next task</u>: Using orbit method, $L^2(\mathbb{R})$ should be a rep of some group H so that the corresponding co-adjoint orbit of H is $\mathbb{R}^2$.

- And the Kirillov character formula should correspond to

$$\mathrm{Tr}(Op(a)) = \int_{\mathbb{R}^2} a(x,\xi)\, dx\, d\xi$$

- The decomposition of $L^2(\mathbb{R})$ and of $\mathbb{R}^2$ reflect our speculation.

$\searrow$ suggests that localised functions (i.e. those in V_i) should be roughly eigenvectors under the action of H.

<u>Q</u>: What is H then?

- $\hat{f}$ being localised is equivalent to $f(x) \approx e^{i\xi_0 x}$?

Why is this true?

$\Rightarrow$ H should contain translation operator

$$(\tau_y f)(t) = f(t+y).$$

$$(\tau_y f)(t) = e^{i\xi(t+y)} = e^{i\xi y} f(t)$$

Take the Fourier transform of the above to get that H should also contain multiplication operator

?

$$(M_y f)(t) = e^{ixy} f(x).$$

$$\widehat{\tau_y(f)}(t) = \tau_y(\hat{f})(t) = \hat{f}(y+t) \xrightarrow{f = e^{i\xi t}} \delta(y+t-\xi)$$

$f = \delta$

$$\delta(y+t-\xi) = \int_{-\infty}^{\infty} e^{-2\pi i (y+t-\xi) z} \, dz$$