Notes for the book Local Langlands for GL(2)

Toan Q. Pham

Friday 24^{th} July, 2020

Contents

Chapter 1

Representations of $GL_n(\mathbb{F}_q)$

Prerequisites

1.1 Some references

Relevant notes:

1. Linear algebraic group: http://www.math.toronto.edu/murnaghan/courses/algp.pdf

2. Rep theory of p-adic groups:

   Check this out http://www.math.tau.ac.il/~bernstei/Publication_list/publication_texts/B-Zel-RepsGL-Usp.pdf

   See youtube lecture: https://youtu.be/jRq1TSiHeHo https://www.youtube.com/user/iiserpunemedia/search?query=langlands https://www.youtube.com/watch?v=r0dhI0vX38s https://sites.google.com/site/repsofpadic/ (see video of talks) (Somehow I can understand this lecture!)

   https://math.mit.edu/~charchan/711LectureNotes.pdf http://virtualmath1.stanford.edu/~conrad/conversesem/refs/snowden.pdf

   http://virtualmath1.stanford.edu/~conrad/conversesem/ New notes found.

   http://people.math.harvard.edu/~gaitsgde/Jerusalem_2010/GradStudentSeminar/p-adic.pdf

   https://webusers.imj-prg.fr/~corinne.blondel/Blondel_Beijin.pdf

   http://www.ims.nus.edu.sg/Programs/liegroups/files/sing.pdf

   http://www.math.lsa.umich.edu/~smdbackr/MATH/notes.pdf

   http://www.math.toronto.edu/murnaghan/courses/mat1197/notes.pdf (The Satake isomorphism, Macdonald's formula and spherical representations)

   https://math.uchicago.edu/~ngo/Rep-p-adic.pdf

   http://www.math.columbia.edu/~phlee/CourseNotes/p-adicGroups.pdf

   https://www.math.toronto.edu/~herzig/smooth_representations.pdf

   http://www.ims.nus.edu.sg/Programs/liegroups/files/Rep-classical-groups.pdf

   https://www.uni-math.gwdg.de/rameyer/download/Guiraud_Diplomarbeit_Jacquets_Functors_in_the_Representation_Theory_of_Reductive_p-Adic_

Groups.pdf Very detailed explanation up to Jacquet functor and parabolic induction.

Godement, IAS, about GL(2) http://www.math.ubc.ca/~cass/research/pdf/godement-ias.pdf

https://www.math.ubc.ca/~cass/research/pdf/Jacquet.pdf

Admissible rep https://www.math.ubc.ca/~cass/research/pdf/p-adic-book.pdf

Summary http://www.math.tifr.res.in/~dprasad/dp-mod-p-2010.pdf

GL2 https://www.math.ucla.edu/~jonr/eprints/padic.pdf

3. Local Langlands for GLn/ GLn rep:

   https://metaphor.ethz.ch/x/2018/fs/401-4114-18L/sc/representations.pdf

   http://web.stanford.edu/~vogti/LLC.pdf

   http://www.math.tifr.res.in/~dprasad/ictp2.pdf (rep of GLn)

   http://www.ma.huji.ac.il/~kazhdan/padic.pdf

4. Automorphic forms and representations: https://math.ou.edu/~kmartin/papers/mfs.pdf

5. Mathematicians:

   Bill Casselman.

6. Some about GL2/n over finite fields:

   https://pdfs.semanticscholar.org/7572/0b6a0f87f340666863a17c43bcf94283f6d1.pdf (last section)

   https://www.imsc.res.in/~amri/html_notes/notes.html

   http://www-users.math.umn.edu/~garrett/m/repns/notes_2014-15/04_finite_GL2.pdf

   https://www.ams.org/journals/tran/1955-080-02/S0002-9947-1955-0072878-2/S0002-9947-1955-0072878-2.pdf

7. http://users.ictp.it/~pub_off/lectures/lns021/Wedhorn/Wedhorn.pdf

8. http://math.columbia.edu/~mundy/llp.html

9. http://math.columbia.edu/~mundy/lcnt.pdf Very short note leading up to Tate's thesis.

10. http://www.math.columbia.edu/~phlee/S17-Langlands/index.html More ref.

11. https://math.columbia.edu/~rdobben/The%20Local%20Langlands%20Correspondences.pdf A essay following C.J. Bushnell, G. Henniart's book.

1.2 Topology

1.2.1 General topology

Definition 1.2.1.1. Given a topological space X and subset S of X. The *subspace topology* on S is defined as: a subset S' of S is open if $S' = S \cap X'$ where X' open subset of X. S is then called *subspace* of X.

Definition 1.2.1.2 (Product topology)**.** The topology of the product $X = \prod_{i \in I} X_i$ of topological spaces X_i whose open sets are union of $\prod_{i \in I} U_i$, where U_i open subsets of X_i with the condition that $U_i = X_i$ for all but finitely many indices i.

Definition 1.2.1.3. A topology is *locally compact* if every point in X has an open neighborhood which is contained in a compact set. A topological space is *compact* if each of its open covers has finite subcover.

Definition 1.2.1.4. A map $f : X \to Y$ is *continuous* if the preimage of any open set is open.

closedmaplemma **Lemma 1.2.1.5.** Every continuous function $f : X \to Y$ from a compact space X to a Hausdorff space Y is closed and proper (i.e. preimages of compact sets are compact).

Continuous image of compact sets are compact. Projections out of product spaces are open maps. More about basic general topology, see this blog. See also Moller's note.

Definition 1.2.1.6. The *closure* of A is the intersection of all closed sets containing A.

Definition 1.2.1.7. A point $x \in X$ is a limit point of A if every neighborhood of x must contain a point in A other than x itself.

1.2.2 Topological groups

Definition 1.2.2.1 (Topological group)**.** A topological group G is a topological space that is also a group such that the group operations of product $G \times G \to G$ and of taking inverses $G \to G$ are continuous (i.e. for any open set $U \subset G$ we have $f^{-1}(U)$ is open in the domain of f, then the function f is continuous). Here $G \times G$ viewed as a topological space with the product topology. So the product is continuous if for all open $U \subset G$ such that $xy \in U$, there exists open $V \times W \subset G \times G$ with $(x, y) \in V \times W$ and $V \cdot W \subset U$, and taking inverse is continuous if for all open $U \subset G$ with $x^{-1} \in G$, there exists open $V \subset G$ with $x \in V$ and $V^{-1} \subset U$.

For a topological group G, the left-multiplication map $h \mapsto gh$ is a homeomorphism. Therefore, in order to describe the topology on G, it is enough to specify a base of open sets about a given element of G, for then one may just translate this base via left-multiplication map to all element of G and get a base for the full topology. For more, see here or here or here or here for some more properties of topological group.

TopGroupProperties **Proposition 1.2.2.2.** For topological group G then

1. Subgroup H of G is also a topological group from the induced topology.
2. Subgroup H is open iff it contains nonempty open subset.
3. Open subgroup H of G is also closed. Furthermore, a closed subgroup H of finite index in G is open.
4. $U \subset G$ open iff gU open iff U^{-1} open iff Ug open.
5. The closure of a subgroup is a subgroup, the closure of a normal subgroup is a normal subgroup.
6. If H subgroup of G then the quotient topology on G/H is defined such as set $U \subset G/H$ is open iff $\rho^{-1}(U)$ is open in the topology of G, where $\rho : G \to G/H$ is the canonical projection map.
7. The canonical projection $\rho : G \to G/H$ is an open map.
8. Each translation map on G/H is continuous function.
9. G/H is Hausdorff iff H closed subgroup of G.
10. G/H discrete topological space iff H open subgroup of G. If furthermore, G is compact then G/H discrete finite topological space iff H is open.
11. If G is Hausdorff, H compact subgroup then canonical projection $\rho : G \to G/H$ is closed map.
12. If $f : G \to H$ is a continuous map of topological groups, then for any closed (res. open) subgroup K of H, $f^{-1}(K)$ is a closed (resp. open) subgroup of H.
13. H subgroup of G, then the closure of H is the intersection of all HU where U open subset containing 1.

Proof. 1.

2. Suppose $U \subset H$ is a nonempty open set then for any $u \in U, h \in H$ we have $hu^{-1}U \subset H$ is an open set (due to left-multiplication being homeomorphism). Hence, $hu^{-1}U \subset H$ is an open neighborhood of h, implying H is open.

3. If H is open subgroup of G, we show H is closed by proving $G \setminus H$ is open. Since H is open so there exists open neighborhood $U \subset H$ of $1 \in H$. For $g \notin H$, note $gU \cap H = \emptyset$ and gU is open, implying $G \setminus H$ is open. Thus, H is closed.

4. This is consequence of the fact that left/right-multiplication and taking inverse are homeomorphisms (being continuous is enough I think).

5. Let $\overline{H}$ the closure of subgroup H of G. If x is a limit point and $h \in H$ then for any neighborhood U of x, we have Uh a neighborhood of xh that contains a point in H other than xh since U contains element in H other than x. Hence, xh is a limit point of H, meaning $xh \in \overline{H}$. Associativity and identity element are inherited from group G. To check for invertible element of limit point x, corresponding to neighborhood U of x we have $U^{-1}x^{-1}$ which is neighborhood of U that contains element in H.

6. Just definition of topology of G/H.

7. Suppose U open map of G, then UH is also open in G. As $\rho^{-1}(\rho(U)) = UH$ and ρ being continuous, we obtain $\rho(U)$ being open in G/H. Thus, ρ is an open map.

8. If U is an open set in G/H then $\bigcup_{g\in U} gH$ is open in G. Say we have left-multiplication ρ'_g by $g' \in G$ then as $\bigcup_{g\in U} g'^{-1}gH$ is open in G so $\rho\left(\bigcup_{g\in U} g'^{-1}gH\right) = (g')^{-1}U$ open in G/H since projection ρ is open map. Thus, left-multiplication by g' on G/H is continuous.

$\square$

Proposition 1.2.2.3. Denote G topological group and $\mathcal{U}$ filter of all neighborhoods of 1.

1. Let $A \subset G$ then $\overline{A} = \bigcap_{U\in\mathcal{U}} AU = \bigcap_{U\in\mathcal{U}} \overline{AU}$.

2. If A is closed and K compact subset of G, then AK is closed subset.

3. For every identity neighborhood U in G there is a closed identity neighborhood C such that $C \subset U$.

1.3 Basic knowledge about local fields

I follow chapter II of Neukirch's algebraic number theory book to learn definition of local field.

There's also a book of I.B. Fesenko and S.V. Vostokov about local fields and their extensions.

1.3.1 Two constructions of field $\mathbb{Q}_p$ of p-adic numbers

Via inverse limit

One view $\mathbb{Z}_p$ as set of formal infinite series $\sum_{v=0}^{\infty} a_v p^v$ (called *p-adic integer*) where $0 \le a_i < p$ for all $i = 0, 1, \ldots$ We then expand $\mathbb{Z}_p$ into formal series $\sum_{v=-m}^{\infty} a_v p^v$

where $m \in \mathbb{Z}$ and $0 \le a_v < p$. Such series are called *p-adic numbers* and write $\mathbb{Q}_p$ set of all these p-adic numbers. If $f \in \mathbb{Q}$ is any rational number, then $f = p^{-m} \cdot g/h$ where $g, h \in \mathbb{Z}$ and $\gcd(gh, p) = 1$. This way, we obtain canonical mapping $\mathbb{Q} \to \mathbb{Q}_p$ which takes $\mathbb{Z}$ to $\mathbb{Z}_p$.

One can turn $\mathbb{Z}_p$ into a ring and $\mathbb{Q}_p$ into its field of fractions. The key is to view $\mathbb{Z}_p$ as $\varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$, i.e. element of $\mathbb{Z}_p$ is viewed as sequence $(s_1, s_2, \ldots)$ where $s_n = \sum_{v=0}^{n-1} a_v p^v \in \mathbb{Z}/p^n\mathbb{Z}$. Addition and multiplication is obtained elementwise in this sequence.

Via p-adic valuation

Representation of p-adic integer

$$a_0 + a_1 p + \cdots, 0 \le a_i < p \tag{1.3.1.1}$$

{eq:p-adicExpansion}

resembles very much the decimal fraction representation $a_0 + a_1\frac{1}{10} + a_2\left(\frac{1}{10}\right)^2 + \cdots, 0 \le a_i < 10$ of a real number between 0 and 10. But it does not converge as the decimal fraction does. The field $\mathbb{Q}_p$ can be constructed from the field $\mathbb{Q}$ in the same fashion as the field of real numbers $\mathbb{R}$. The key to this is replace the ordinary absolute value by a new p-adic absolute value $||_p$ with respect to which the series eq. (1.3.1.1) converge so that the p-adic numbers appear in the usual manner as limits of Cauchy sequences of rational numbers.

The p-adic absolute value $||_p$ is defined as follow: Let $a \in \mathbb{Q}$ be nonzero rational number. We can write $a = p^m \cdot (b/c)$ where $\gcd(bc, p) = 1$ and we put $|a|_p = p^{-m}$. The exponent m in the representation of a is denoted by $v_p(a)$, and one put formally $v_p(0) = \infty$. This gives the function $v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$, which can be checked to satisfy following property

1. $v_p(a) = \infty \iff a = 0$,
2. $v_p(ab) = v_p(a) + v_p(b)$,
3. $v_p(a+b) \ge \min\{v_p(a), v_p(b)\}$.

The function v_p is called the *p-adic exponential valuation* of $\mathbb{Q}$. The *p-adic absolute value* is given by

$$||_p : \mathbb{Q} \to \mathbb{R}, a \mapsto |a|_p = p^{-v_p(a)}$$

which satisfies the conditions of a *norm* on $\mathbb{Q}$:

1. $|a|_p = 0 \iff a = 0$,
2. $|ab|_p = |a|_p|b|_p$,
3. $|a+b|_p \le \max\{|a|_p, |b|_p\} \le |a|_p + |b|_p$.

One will see later that the absolute values $||_p$ and $||$ essentially exhaust all norms on $\mathbb{Q}$.

Now, we give new definition of field $\mathbb{Q}_p$. A *Cauchy sequence* with respect to $||_p$ is a sequence $\{x_n\}$ of rational numbers such that for every $\varepsilon > 0$, there exists a positive integer n_0 satisfying $|x_n - x_m| < \varepsilon$ for all $n, m \geq n_0$. A sequence $\{x_n\}$ in $\mathbb{Q}$ is called a *nullsequence* with respect to $||_p$ if $|x_n|_p$ is a sequence converging to 0 in the usual sense.

The Cauchy sequences form a ring R, the nullsequences form a maximal ideal $\mathfrak{m}$, and we define $\mathbb{Q}_p := R/\mathfrak{m}$. We embed $\mathbb{Q}$ in $\mathbb{Q}_p$ by associating to every element $a \in \mathbb{Q}$ the residue class of the constant sequence $(a, a, \ldots)$. The p-adic absolute value $||_p$ on $\mathbb{Q}$ is extended to $\mathbb{Q}_p$ by giving element $x = \{x_n\} \mod \mathfrak{m} \in R/\mathfrak{m}$ the absolute value

$$|x|_p := \lim_{n\to\infty} |x_n|_p \in \mathbb{R}.$$

This limit exists because $\{|x_n|_p\}$ is a Cauchy sequence in $\mathbb{R}$ ($||x_n|_p - |x_m|_p| \leq |x_n - x_m|_p < \varepsilon$), and it is independent of the choice of the sequence $\{x_n\}$ within its class mod $\mathfrak{m}$ because any p-adic nullsequence $\{y_n\} \in \mathfrak{m}$ satisfies $\lim_{n\to\infty} |y_n|_p = 0$.

The p-adic exponential value v_p on $\mathbb{Q}$ extends to an exponential valuation $v_p : \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$ as follow: If $x \in \mathbb{Q}_p$ is the class of Cauchy sequence $\{x_n\}$ where $x_n \neq 0$, then $v_p(x_n) = -\log_p |x_n|_p$ either diverges to ∞ or is a Cauchy sequence in $\mathbb{Z}$ which eventually must become constant for large n since $\mathbb{Z}$ is discrete. Hence, we put

$$v_p(x) = \lim_{n\to\infty} v_p(x_n) = v_p(x_n) \text{ for } n \geq n_0.$$

The field of $\mathbb{Q}_p$ is complete with respect to $||_p$, i.e. every Cauchy sequence in $\mathbb{Q}_p$ converges with respect to $||_p$.

Due to an important property of $||_p$, which is $|x+y| \leq \max\{|x|_p, |y|_p\}$, we obtain the fact that the set $\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ is a subring of $\mathbb{Q}_p$ and is the closure with respect to $||_p$ of the ring $\mathbb{Z}$ in $\mathbb{Q}_p$.

The group of units of $\mathbb{Z}_p$ is $\mathbb{Z}_p^* = \{x \in \mathbb{Z}_p : |x|_p = 1\}$. Every element $x \in \mathbb{Q}_p^*$ admits a unique representation $x = p^m u$ with $m \in \mathbb{Z}, u \in \mathbb{Z}_p^*$. Indeed, if $v_p(x) = m \in \mathbb{Z}$ then $v_p(xp^{-m}) = 0$, hence $|xp^{-m}|_p = 1$, i.e. $u = xp^{-m} \in \mathbb{Z}_p^*$. Furthermore, we find that the nonzero ideals of the ring $\mathbb{Z}_p$ are the principal ideal $p^n\mathbb{Z}_p = \{x \in \mathbb{Q}_p : v_p(x) \geq n\}$, with $n \geq 0$ and one has $\mathbb{Z}_p/p^n\mathbb{Z}_p \cong \mathbb{Z}/p^n\mathbb{Z}$. With this, we can show that $\mathbb{Z}_p$ is exactly the same as in first definition, i.e. $\mathbb{Z}_p \cong \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$.

1.3.2 Valuations

We generalize the previous procedure of producing $\mathbb{Q}_p$ from $\mathbb{Q}$ using the concept of valuation:

Definition 1.3.2.1. A *valuation* of a field K is a function $|.| : K \to \mathbb{R}$ satisfying following

1. $|x| \geq 0$ and $|x| = 0 \iff x = 0$,
2. $|xy| = |x||y|$,
3. $|x+y| \leq |x| + |y|$

If the stronger condition $|x+y| \leq \max(|x|, |y|)$ holds, then $|.|$ is *nonarchimedean*.

With this, one can turn K into a metric space (in paricular a Hausdorff topological space) by defining distnce between $x, y \in K$ by $d(x,y) = |x-y|$. Two valuations of K are called *equivalent* if they denote the same topology on K. One can show that two valuations $||_1$ and $||_2$ on K are equivalent iff there exists real number $s > 0$ such that $|x|_1 = |x|_2^s$ for all $x \in K$.

We then have approximation theorem (but I guess it is irrelevant at this point to write out).

The valuation $||$ is called *nonarchimedean* if $|n|$ stays bounded for all $n \in \mathbb{N}$. Equivalently, it is nonarchimedean iff the inequality $|x+y| \leq \max\{|x|, |y|\}$.

Let $||$ be nonarchimedean valuation of field K. Putting

$$v(x) = -\log|x| \text{ for } x \neq 0 \text{ and } v(0) = \infty,$$

we obtain a function $v : K \to \mathbb{R} \cup \{\infty\}$ satisfying the same properties for v_p of $\mathbb{Q}_p$. Function v on K with these properties is called *exponential valuation* on K. For any exponential valuation v we obtain a valuation (which will be called *multiplicative valuation* or *absolute value*) $||$ by putting $|x| = q^{-v(x)}$ for some fixed real $q > 1$.

Example 1.3.2.2 (Valuation of rational function field)**.** Let $k(t)$ rational function field over k, we have the ring $k[t]$ inside $k(t)$, the prime ideal $\mathfrak{p} \neq 0$ which are given by monic irreducible polynomial $p(t) \in k[t]$. For every such $\mathfrak{p}$, one defines absolute value $||_\mathfrak{p} : k(t) \to \mathbb{R}$ as follow: Let $f(t) = g(t)/h(t), g(t), h(t) \in k[t]$ be nonzero rational function. We extract from $g(t), h(t)$ the highest possible power of irreducible polynomial $p(t)$, say power is m, and put

$$v_\mathfrak{p}(f) = m, |f|_\mathfrak{p} = q_\mathfrak{q}^{-v_\mathfrak{p}(f)}.$$

where $q_\mathfrak{p} = q^{d_\mathfrak{p}}$, $d_\mathfrak{p}$ being the degree of residue class field of $\mathfrak{p}$ over k and q fixed real number > 1. Furthermore, we put $v_\mathfrak{p}(0) = \infty$ and $|0|_\mathfrak{p} = 0$.

For function field $k(t)$, there is one more exponential valuation $v_\infty : k(t) \to \mathbb{Z} \cup \infty$, namely $v_\infty(f) = \deg h - \deg g$, where $f = g/h \neq 0, g, h \in k[t]$. ⌟

Example 1.3.2.3 (Valutation of $\mathbb{Q}_p$)**.** We have $v_p(0) = 0$ and for any $x \in \mathbb{Q}_p^\times$ then exists $m \in \mathbb{Z}, u \in \mathbb{Z}_p$ so $x = p^m u$. Let $v_p(x) = -m$. ⌟

The conditions for v imply that

Proposition 1.3.2.4. The subset $\mathcal{O} = \{x \in K : v(x) \geq 0\} = \{x \in K : |x| \leq 1\}$ is a ring with group of units $\mathcal{O}^* = \{x \in K : v(x) = 0\} = \{x \in K : |x| = 1\}$ and the unique maximal ideal $\mathfrak{p} = \{x \in K : v(x) > 0\} = \{x \in K : |x| < 1\}$.

$\mathcal{O}$ is an integral domain with field of fractions K and has the property that for every $x \in K$, at least one of x, x^{-1} must be in $\mathcal{O}$. Such a ring is called *valuation ring*. Its only maximal ideal is $\mathfrak{p} = \{x \in \mathcal{O} : x^{-1} \notin \mathcal{O}\}$. The field $\mathcal{O}/\mathfrak{p}$ is called the *residue class field* of $\mathcal{O}$. A valuation ring is always integrally closed.

An exponential valuation ring v is called *discrete* if it admits a smallest positive value s. In this case, one finds $v(K^*) = s\mathbb{Z}$.

Question 1.3.2.5. Why having s as smallest positive value would imply $v(K^*) = s\mathbb{Z}$.

It is called *normalized* if $s = 1$. Dividing by s we may always pass to normalized valuation without changing the invariants $\mathcal{O}, \mathcal{O}^*, \mathfrak{p}$. Having done so, an element $\pi \in \mathcal{O}$ such that $v(\pi) = 1$ is a *prime element*, and every element $x \in K^*$ admits unique representation $x = u\pi^m$ with $m \in \mathbb{Z}$ and $u \in \mathcal{O}^*$. For if $v(x) = m$, then $v(x\pi^{-m}) = 0$, hence $u = x\pi^{-m} \in \mathcal{O}^*$.

Proposition 1.3.2.6. If v discrete exponential valuation of K, then $\mathcal{O} = \{x \in K : v(x) \geq 0\}$ is a principal ideal domain, hence a discrete valuation ring (i.e. a principal ideal domain with unique maximal ideal). Suppose v is normalized. Then the nonzero ideals of $\mathcal{O}$ are given by

$$\mathfrak{p}^n = \pi^n\mathcal{O} = \{x \in K : v(x) \geq n\},\ n \geq 0,$$

where π is a prime element, i.e. $v(\pi) = 1$. One has $\mathfrak{p}^n/\mathfrak{p}^{n+1} \cong \mathcal{O}/\mathfrak{p}$.

Question 1.3.2.7. Why $\mathfrak{p}^n = \pi^n\mathcal{O}$?

Answer. Since v is discrete exponential normalized valuation of K so $v(K^*) = \mathbb{Z}$. Hence, if $v(x) > 0$ means $v(x) \geq 1$. This follows from the definition of $\mathfrak{p}$ that $\mathfrak{p} = \{x \in K : v(x) \geq 1\}$ so $\pi \in \mathfrak{p}$. Note $v(\pi^{-1}) = -v(\pi) = -1$ so $v(\mathfrak{p}\pi^{-1}) \geq 0$ implying $\mathfrak{p}\pi^{-1} \subset \mathcal{O}$. Conversely, $\mathcal{O} \subset \mathfrak{p}\pi^{-1}$ due to valutation v. $\square$

In a discrete valued field K the chain $\mathcal{O} \supset \mathfrak{p} \supset \mathfrak{p}^2 \supset \mathfrak{p}^3 \supset \cdots$ consisting of the ideals of the valuation ring $\mathcal{O}$ forms a basis of neighborhoods of the zero element. Indeed, if v is normalozed exponential valuation and $|| = q^{-v}(q > 1)$ an associated multiplicative valuation, then

$$\mathfrak{p}^n = \left\{x \in K : |x| < q^{1-n}\right\}.$$

Question 1.3.2.8. Why $\mathfrak{p}^n$ is like this? Shouldn't it be $|x| \leq q^{-n}$?

Answer. Note we are talking about discrete normalized valuation v so $v(K^*) = \mathbb{Z}$, so $v(x) \geq n$ equivalent to $v(x) > n - 1$ which gives $|x| < q^{1-n}$. $\square$

Example 1.3.2.9. So here we have $K = \mathbb{Q}_p, \mathcal{O} = \mathbb{Z}_p$ and $\mathfrak{p}^n = p^n\mathbb{Z}_p$. ⌟

Example 1.3.2.10. In case of rational function field $K = k(t)$, say $\mathfrak{p} = (x - a)$ then with $v_{\mathfrak{p}}$ as defined before, we find $\mathcal{O}$ consists of rational functions $f(t) = (x-a)^m \cdot g(t)/h(t)$ where $m \geq 0$. The residue class field then is just k. ⌟

As a basis of neighborhoods of element 1 of K^*, we obtain in the same way the descending chain $\mathcal{O}^* = U^{(0)} \supset U^{(1)} \supset U^{(2)} \supset \cdots$ of subgroups

$$U^{(n)} = 1 + \mathfrak{p}^n = \left\{ x \in K : |1 - x| < q^{1-n} \right\}, n > 0,$$

of $\mathcal{O}^*$. $U^{(n)}$ is called the nth *higher unit group* and $U^{(1)}$ the group of *principal units*.

Proposition 1.3.2.11. $\mathcal{O}^*/U^{(n)} \cong (\mathcal{O}/\mathfrak{p}^n)^*$ and $U^{(n)}/U^{(n+1)} \cong \mathcal{O}/\mathfrak{p}$, for $n \geq 1$.

1.3.3 Completions

Definition 1.3.3.1. A valued field $(K, ||)$ is called *complete* if every Cauchy sequence $\{a_n\}$ in K converges to an element $a \in K$, i.e. $\lim_{n\to\infty} |a_n - a| = 0$.

From any valued field $(K, ||)$ we get a complete valued field $(\hat{K}, ||)$ by process of *completion*. Take ring R of all Cauchy sequence of $(K, ||)$, consider maximal ideal $\mathfrak{m}$ of all nullsequences with respect to $||$, and define $\hat{K} = R/\mathfrak{m}$. One emebed the field K into $\hat{K}$ by sending every $a \in K$ to $(a, a, \ldots)$. The valuation $||$ is extended from K to $\hat{K}$ by giving $a = \{a_n\} \in \hat{K}$ the absolute value $|a| = \lim_{n\to\infty} |a_n|$. The limit exists because $\{|a_n|\}$ is a Cauchys sequence. One then proves $\hat{K}$ is complete with respect to the extended $||$.

Fields $\mathbb{R}$ and $\mathbb{C}$ are example of complete fields with respect to an archimedean valuation. There are no other of this type, due to following

Theorem 1.3.3.2 (Ostrowski)**.** Let K be a field which is complete with respect to an archimedean valuation $||$. Then there is an isomorphism σ from K to $\mathbb{R}$ or $\mathbb{C}$ satisfying $|a| = |\sigma a|^s$ for all $a \in K$, for some fixed $s \in (0, 1]$.

Hence, we will restrict attention to nonarchimedean valuations. Let v be exponential valuation of such field K. It is canonically continued to an exponential valuation $\hat{v}$ of the completion $\hat{K}$ by setting $\hat{v}(a) = \lim_{n\to\infty} v(a_n)$ where $a = \{a_n\}$. Observe that sequence $v(a_n)$ has to become stationary (provided $a \neq 0$) because one has $\hat{v}(a - a_n) > \hat{v}(a)$, so that it follows from the remark on p. 119 that

$$v(a_n) = \hat{v}(a_n - a + a) = \min\{\hat{v}(a_n - a), \hat{v}(a)\} = \hat{a}.$$

Question 1.3.3.3. Why $\hat{v}(a - a_n) > \hat{v}(a)$

It therefore follows that $v(K^*) = \hat{v}(\hat{K}^*)$, and if v is discrete and normalized, then so is the extension $\hat{v}$.

Proposition 1.3.3.4. If $\mathcal{O} \subset K$, resp $\widehat{\mathcal{O}} \subset \widehat{K}$, is the valuation ring of v, resp. of $\widehat{v}$, and $\mathfrak{p}$, resp. $\widehat{\mathfrak{p}}$, is the maximal ideal, then one has $\widehat{\mathcal{O}}/\widehat{\mathfrak{p}} \cong \mathcal{O}/\mathfrak{p}$ and, if v is discrete, one has furthermore $\widehat{\mathcal{O}}/\widehat{\mathfrak{p}}^n \cong \mathcal{O}/\mathfrak{p}^n$ for $n \geq 1$.

Example 1.3.3.5. With $K = \mathbb{Q}$, using the p-adic absolute value $||_p$ we find valuation ring of $\mathbb{Q}$ is the local ring $\mathcal{O} = \mathbb{Z}_{(p)} = \left\{\frac{z}{n} : z, n \in \mathbb{Z}, p \nmid n\right\}$.

Taking the completion of $\mathbb{Q}$ with respect to $||_p$ gives $\mathbb{Q}_p$ and we know $\widehat{\mathcal{O}} = \mathbb{Z}_p$ and $\widehat{\mathfrak{p}} = p\mathbb{Z}_p$. Hence, from above proposition, we obtain

$$\mathbb{Z}_p/p^n\mathbb{Z}_p \cong \mathbb{Z}_{(p)}/p^n\mathbb{Z}_{(p)} \cong \mathbb{Z}/p^n\mathbb{Z}.$$

This gives us proposition (2.4) in the book. ⌟

Following proposition generalizing p-adic expansion to any discrete valutation v of K:

Proposition 1.3.3.6. Let $R \subset \mathcal{O}$ system of representatives for $\kappa = \mathcal{O}/\mathfrak{p}$ such that $0 \in R$, and let $\pi \in \mathcal{O}$ be a prime element. Then every $x \neq 0$ in $\widehat{K}$ admits a unique representation as a convergent series

$$x = \pi^m(a_0 + a_1\pi + \dots)$$

where $a_i \in R, a_0 \neq 0, m \in \mathbb{Z}$.

Example 1.3.3.7. In case of $K = \mathbb{Q}$ and p-adic valuation v_p, we have $\widehat{K} = \mathbb{Q}_p$, the numbers $0, 1, \dots, p-1$ form a system of representatives R for residue class field $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$ of the valuation, and we get back representation of p-adic numbers. ⌟

Example 1.3.3.8. In case of $K = k(t)$ and valuation $v_{\mathfrak{p}}$ attached to prime ideal $\mathfrak{p} = (t-a)$ of $k[t]$, we take system of representatives R of the field of coefficients of k itself. The completion is the *field of formal power series* $k((x))$, $x = t - a$, consisting of all formal Laurent series

$$f(t) = (t-a)^m(a_0 + a_1(t-a) + a_2(t-a)^2 + \cdots),$$

with $a_i \in k$ and $m \in \mathbb{Z}$. ⌟

Proposition 1.3.3.9. There is a canonical isomorphism $\mathcal{O} \to \varprojlim_n \mathcal{O}/\mathfrak{p}^n$ and is also a homeomorphism. The same is true for the mapping $\mathcal{O}^* \to \varprojlim_n \mathcal{O}^*/U^{(n)}$.

Next, we concern about finite extension L/K of a complete valued field.

Theorem 1.3.3.10. Let K be complete with respect to valuation $||$. Then $||$ may be extended in a unique way to obtain a valuation of any given algebraic extension L/K. This extension is given by formula $|\alpha| = \sqrt[n]{|N_{L/K}(\alpha)}$ when L/K has finite degree n. In this case L is again complete.

1.3.4 Local fields

Local fields are fields with respect to a discrete valuation and have a finite residue class field. For such local fields, the normalized exponential valuation is denoted by $v_{\mathfrak{p}}$, and $||_{\mathfrak{p}}$ denotes the absolute value normalized by $|x|_{\mathfrak{p}} = q^{-v_{\mathfrak{p}}(x)}$, where q is the cardinality of the residue class field.

Proposition 1.3.4.1. A local field K is locally compact. Its valuation ring $\mathcal{O}$ is compact.

Question 1.3.4.2. There is an argument in this proof saying that $a + \mathcal{O}$ is open and compact neighborhood of a in K. Is this true it is open while $\mathcal{O}$ is closed. Also, I.B. Fesenko and S.V. Vostokov's book claim that with the topology of K induced by discrete valuation v (page 7) then $\alpha + \pi^n\mathcal{O}, n \in \mathbb{Z}$ form a basis of open neighborhoods of α. Why open?

Answer. Note we are talking about discrete valuation v, meaning $v(K^*) = \mathbb{Z}$, so $\mathcal{O} = \{x \in K : v(x) \geq 0\}$ is the same as $\{x \in K : v(x) > -1\} = \{x \in K : |x| < q^{-1}\}$ which is open. Hence $a + \mathcal{O}$ is open.

Similarly, $\pi^n\mathcal{O}$ for $n \in \mathbb{Z}$ is the set $\{x \in K : v(x) \geq n\} = \{x \in K : v(x) > n - 1\}$ and hence is open. $\square$

Proposition 1.3.4.3. The local fields are precisely finite extensions of the fields $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$.

Hence, the local fields of characteristic p are the *power series fields* $\mathbb{F}_q((t))$ with $q = p^f$. The local fields of characteristic 0, i.e. finite extensions $K/\mathbb{Q}_p$, are called $\mathfrak{p}$-*adic number fields*.

Chapter 2

Smooth representations

http://www.math.wm.edu/~vinroot/PadicGroups/

2.1 Locally profinite groups

2.1.1 Definition

Definition 2.1.1.1. Directed partially ordered set, inverse limit, ...

A partially ordered set I is called *direct set* if for any $i, j \in I$, there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

Definition 2.1.1.2. Let $\mathcal{G}$ be category of finite groups and $\mathcal{J}$ be an filtered index category (i.e. a directed partially ordered set I). A *profinite group* is the limit (also called inverse limit or projective limit) of the diagram $F : \mathcal{J} \to \mathcal{G}$, where $F(i) = G_i$. We write $G = \varprojlim_i G_i$.

As a set, we have

$$G = \left\{ (g_i)_{i \in I} \prod_{i \in I} G_i : \varphi_{ij}(g_i) = g_j \; i \geq j \right\}.$$

If each G_i is assumed to be discrete topology and $\varphi_{i,j}$ as continuous homomorphisms of groups then G is a topological group, , a subspace topology from product topology $\prod G_i$. Also $\varphi_i : G \to G_i$ are continuous homomorphisms of groups.

Due to G_i being finite, so $G \subset \prod_{i \in I} G_i$ is closed, and since $\prod_{i \in I} G_i$, as the product space of compact space, is still compact, G is also compact. One can also see that G is totally disconnected (i.e. connected components of G are point-sets). Hence, any profinite group is compact and totally disconnected. As a consequence, all open subgroups of profinite group are closed, and a closed subgroup is open iff it is of finite index.

Exercise 1. Show G is a topological group with corresponding requirement. Explain more.

Proposition 2.1.1.3. Let $\mathcal{C}$ be formation of finite groups (i.e. image of $F(\mathcal{J})$). Then the following condition on a topological group G are equivalent:

1. G is a profinite group with $\mathcal{C}$ as set of finite groups;

2. G is compact Hausdorff totally disconnected, and for each open normal subgroup U of G, $G/U \in \mathcal{C}$;

3. The identity element 1 of G admits system $\mathcal{U}$ of open neighborhoods U such that each U normal subgroup of G with $G/U \in \mathcal{C}$, and $G = \varprojlim_{U\in\mathcal{U}} G/U$.

Proof. See theorem 2.1.3 of book Profinite groups. Or theorem §2, chapter 4 of Neurkind algebraic number theory book. Some links: MSE □

Definition 2.1.1.4. A *locally profinite group* is a topological group G such that every open neighborhood of the identity in G contains a compact open subgroup of G.

Any discrete group is locally profinite. A closed subgroup of a locally profinite group is locally profinite. The quotient of a locally profinite group by a closed normal subgroup is locally profinite.

The relation between definition of locally profinite group and of profinite group is as follow: Locally profinite group G is isomorphic to $\varprojlim G/K$ where K ranges over open normal subgroups of G. A (locally) profinite group is (locally) compact and totally profinite. Conversely, a (locally) compact, totally disconnected group is (locally) profinite.

Remark 2.1.1.5. Read more at here or the book Profinite groups by Luis Ribes.

Exercise 2. Proof above statement.

Exercise 3. Let G be a locally profinite group, and let H be a closed subgroup of G.Thus H is also locally profinite.

2.1.2 Non-Archimedean local field F is locally profinite

subsection1.2

F is non-Archimedean local field. Thus, F is field of fractions of discrete valuation ring $\mathfrak{o}$. Let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$ and $\mathbf{k} = \mathfrak{o}/\mathfrak{p}$ be the residue class field. We assume that $\mathbf{k}$ is finite, and we denote cardinality of $\mathfrak{k}$ to be q.

Since $\mathbf{k}$ is finite so $\mathfrak{p}^n$ and $\mathfrak{o}$ are compct as argued in the book.

2.1.3 $F^\times$ is locally profinite

2.1.4 $M_n(F)$ and $GL_n(F)$ are locally profinite

2.1.5 $\mathfrak{o}$-lattice and lattice of F-vector space

section1.5

Let V be F-vector space of finite dimension n. An *$\mathfrak{o}$-lattice* in V is a finitely generated $\mathfrak{o}$-submodule L of V such that the F-linear span FL of L is V.

Proposition 2.1.5.1. Let L be $\mathfrak{o}$-lattice of V. There is an F-basis $\{x_1, \dots, x_n\}$ of V such that $L = \sum_{i=1}^n \mathfrak{o} x_i$.

Proof. Same as in the book. One can multiply both side of $\sum a_i y_i = 0$ by element of $F^\times$ to assume that all $a_i \in \mathfrak{o}$ and that one of them, a_j say, is a unit of $\mathfrak{o}$. Indeed, at the beginning, we know that not all a_i's are zero. For nonzero a_i, i.e. $a_i \in F^\times$, from §2.1.2, one can write $a_i = u\varpi^n$ for $u \in U_F, n \in \mathbb{Z}$. Note that if $n < 0$ then $a_i \notin \mathfrak{o}$, if $n \geq 0$ then $a_i \in \mathfrak{o}$ so one can find the smallest $n < 0$ and multiply both side by ϖ^{-n}, which will give what we want. □

In particular, the $\mathfrak{o}$-lattice L is a compact open subgroup of V. The $\mathfrak{o}$-lattice in V give a fundamental system of open neighborhoods of 0 in V.

Explain. We know $V \cong F^n$ via F-basis $\{x_1, \dots, x_n\}$ given in previous proposition. This gives a topology for V. Note $L = \sum_{i=1}^n \mathfrak{o} x_i$ from previous proposition so L is identified with $\mathfrak{o}^n$. Note that $\mathfrak{o}$ is compact so L is compact. Furthermore, also from $L = \sum_{i=1}^n \mathfrak{o} x_i$, L is a subgroup of F since $\mathfrak{o}$ is a subring of F. Note $\mathfrak{o}$ is also open since we are talking about discrete valuation v. □

More generally, a *lattice* in V is a compact open subgroup of V. Here we have

Lemma 2.1.5.2. Let L subgroup of V. Then L is a lattice in V iff there exists $\mathfrak{o}$-lattices L_1, L_2 in V such that $L_1 \subset L \subset L_2$.

Proof. Suppose $L_1 \subset L \subset L_2$, where L_i are $\mathfrak{o}$-lattices. Since L contains L_1 so it is open and hence closed (for any $x \in L$ then $x + L_1 \subset L$ an open neighborhood of x in L, implying L is open; for any limit point $x \in V$ of L, open set $x + L_1$ must contain some element in L, implying $x \in L$, hence L is closed). Since closed L is contained in compact L_2, it is compact. Thus, L is a lattice in V.

Conversely, if L is a lattice in V, it must contain an $\mathfrak{o}$-lattice (since L open neighborhood of 0 since L subgroup), and so $FL = V$. In the opposite direction, we choose a basis $\{x_1, \dots, x_n\}$ of V. The image of L under obvious projection $V \to Fx_i$ is a compact open subgroup of Fx_i (since projection is an open map, and continuous map sends compact to compact, the projection is also group homomorphism). It is therefore contained in a group of the form $\mathfrak{a}_i x_i$, for some fractional ideal $\mathfrak{a}_i = \mathfrak{p}_i^{a_i}$ of $\mathfrak{o}$. (OK I stuck this part but let's just skip it for now) □

2.1.6 Characters of locally profinite group

section1.6

Let G locally profinite group.

Proposition 2.1.6.1. Let $\psi : G \to \mathbb{C}^\times$ be group homomorphism. The following are equivalent:

1. ψ is continuous.

2. the kernel of ψ is open.

If φ satisfies these conditions and G union of its compact open subgroups, then the image of ψ is contained in the unit circle $|z| = 1$ in $\mathbb{C}$.

Proof. (2) $\implies$ (1) since preimage of $f(g)$ in G is $g + \ker\psi$, which is open and also union of any number of open sets is open. Conversely, let $\mathcal{N}$ open neighborhood of 1 in $\mathbb{C}$. Thus $\psi^{-1}(\mathcal{N})$ is open containing identity of G so it contains a compact open subgroup K of G (since G is locally profinite group). However, if $\mathcal{N}$ is chosen sufficiently small, it contains no non-trivial subgroup of $\mathbb{C}^\times$ (say $\mathcal{N}$ contains some non-trivial subgroup of $\mathbb{C}^\times$ which contains some $z = re^h$ then for sufficiently larger n, $z^n = r^n e^{hn}$ can be very far from 1 out of $\mathcal{N}$, a contradiction). Thus, $K \subset \ker\psi$ which follows $\ker\psi$ is open.

The unit circle S^1 is the maximal compact subgroup of $\mathbb{C}^\times$. If K is a compact subgroup of G, then $\psi(K)$ is compact so it is contained in S^1. □

We define a *character* of locally profinite group G to be continuous homomorphism $G \to \mathbb{C}^\times$. We usually write 1_G, or even just 1, for the trivial (constant) character of G. We call a character *unitary* if its image is contained in the unit circle.

2.1.7 Characters of F

section1.7

Set of characters of F is a group under multiplication, we denote it $\widehat{F}$. Since F is union of its compact open subgroups $a + \mathfrak{p}^n, n \in \mathbb{Z}$, all characters of F are unitary.

If $\psi \neq 1$ is a character, then there is least integer d such that $\mathfrak{p}^d \subset \ker\psi$ (due to definition of $\mathfrak{p}^d$).

Definition 2.1.7.1. Let $\psi \in \widehat{F}, \psi \neq 1$. The *level* of ψ is the least integer d such that $\mathfrak{p}^d \subset \ker\psi$.

If we fix d, the set of characters of F of level $\leq d$ is the subgroups of $\psi \in \widehat{F}$ such that $\psi|_{\mathfrak{p}^d} = 1$.

Proposition 2.1.7.2 (Additive duality)**.** Let $\psi \in \widehat{F}, \psi \neq 1$, have level d.

1. Let $a \in F$. The map $a\psi : x \mapsto \psi(ax)$ is a character of F. If $a \neq 0$, the character $a\psi$ has level $d - v_F(a)$.

2. The map $a \mapsto a\psi$ is a group isomorphism $F \cong \widehat{F}$.

Proof. Showing $a\psi$ is character is obvious. Note that $a\mathfrak{p}^{d-v_F(a)} \subset \mathfrak{p}^d \subset \ker\psi$ so the level of $a\psi$ must be at most $d - v_F(a)$. To do that, we must show there exists $o\varpi^{d-v(a)-1} \in \mathfrak{p}^{d-v(a)-1}$ such that $\psi(ao\varpi^{d-v(a)-1}) \neq 1$. Indeed, first as $a \neq 0$, we write $a = u\varpi^{v(a)}$ with $u \in \mathfrak{o}^\times$. Hence, $ao\varpi^{d-v(a)-1} = uo\varpi^{d-1}$. On the other hand, since d level of ψ so exists $b = u'\psi^{d-1} \in \mathfrak{p}^{d-1}$ such that $\psi(b) \neq 1$ where $u' \in \mathfrak{o}^\times$. Hence, this suggests to choose o such that $uo = u'$. Thus, $d - v_F(a)$ is the level of $a\psi$.

Why map $a \mapsto a\psi$ is an injective group homomorphism?

Let $\theta \in \widehat{F}, \theta \neq 1$ and let l be level of θ. Let ϖ be prime element of F, and $u \in U_F$. The character $u\pi^{d-l}\psi$ has level l, and so agree with θ on $\mathfrak{p}^l$. The characters $u\varpi^{d-l}\psi, u'\varpi^{d-l}\psi, u, u' \in U_F$, agree on $\mathfrak{p}^{l-1}$ if and only if $u \equiv u' \pmod{\mathfrak{p}}$ Stuck □

Exercise 4. Let L be a lattice in F and χ be character of L. Show that there exists a character ψ of F such that $\psi|_L = \chi$.

2.1.8 Characters of $F^\times$

section1.8

We turn to the multiplicative group $F^\times$. Let χ be a character of $F^\times$. By argument in proposition 2.1.6, χ is trivial on $U_F^m = 1 + \mathfrak{p}^m$ for some $m \geq 0$:

Definition 2.1.8.1. Let χ be non-trivial character of $F^\times$. The *level* of χ is defined to be least integer $n \geq 0$ such that χ is trivial on U_F^{n+1}.

We use the same terminology for characters of open subgroups of $F^\times$. Observe that a character of $F^\times$ need not to be unitary: for example, the map $x \mapsto \|x\|$ is a character. Note also that, in contrast to the additive case, $F^\times$ has a *unique maximal compact subgroup*, namely U_F.

Question 2.1.8.2. Why U_F is unique maximal compact subgroup of $F^\times$?

The structure of the group of characters of $F^\times$ is more subtle than that of $\widehat{F}$. However, we shall make frequent use of a partial description in additive terms. Let m, n be integers, $1 \leq m < n \leq 2m$. The map $x \mapsto 1 + x$ gives an isomorphism $\mathfrak{p}^m/\mathfrak{p}^n \cong U_F^m/U_F^n$. Indeed, it is a group homorphism as for $x, y \in \mathfrak{p}^m$, since $2m \geq n$ so $xy \equiv 0 \pmod{\mathfrak{p}^n}$, implying

$$x + y \pmod{\mathfrak{p}^n} = x + y + xy \pmod{\mathfrak{p}^n} \mapsto 1 + x + y + xy = (1+x)(1+y) \pmod{U_F^n}$$

It is certainly bijective. Hence, this gives an isomorphism of character of groups $(\widehat{\mathfrak{p}^m/\mathfrak{p}^n}) \cong (\widehat{U_F^m/U_F^n})$. And we can use 2.1.7 to describe the group $(\widehat{\mathfrak{p}^m/\mathfrak{p}^n})$. Fix a character $\psi_F \in \widehat{F}$ of level 1. For $a \in F$, we define a function

$$\psi_{F,a} : F \to \mathbb{C}^\times, \psi_{F,a}(x) = \psi_F(a(x-1)).$$

Proposition 2.1.7 then yields:

Proposition 2.1.8.3. Let $\psi \in \widehat{F}$ have level 1. Let m, n be integers, $0 \leq m < n \leq 2m + 1$. The map $a \mapsto \psi_{F,a}|_{U_F^{m+1}}$ induces an isomorphism

$$\mathfrak{p}^{-n}/\mathfrak{p}^{-m} \to \widehat{U_F^{m+1}/U_F^{n+1}}.$$

Observe that, viewed as character of U_F^{m+1}, the function $\psi_{F,a}$ has level $-v_F(a)$. Also, the condition relating m and n can be re-formulated as $\lfloor n/2 \rfloor \leq m < n$.

Question 2.1.8.4. Explain above proposition. How do we exactly describe $\widehat{\mathfrak{p}^m/\mathfrak{p}^n}$ from proposition 2.1.7?

2.2 Smooth representations of locally profinite groups

2.2.1 Smooth representation

section2.1

Let G locally profinite group, and (π, V) representation of G. Thus, V complex vector space and π group homomorphism $G \to \mathrm{Aut}_{\mathbb{C}}(V)$. The representation (π, V) is called *smooth* if, for every $v \in V$, there is a compact open subgroup K of G (depending on v) such that $\pi(x)v = v$, for all $x \in K$. Equivalently, if V^K denotes the space of $\pi(K)$-fixed vectors in V, then $V = \bigcup_K V^K$ where K ranges over all compact open subgroups of G.

A smooth representation (π, V) is called *admissible* if the space V^K is finite-dimensional for each compact open subgroup K of G.

Let (π, V) be a smooth representation of G, then any G-stable subspace of G provides a further smooth representation of G. Likewise, if U is a G-subspace of V, the natural representation of G on the quotient V/U is smooth. One says that (π, V) is *irreducible* if $V \neq 0$ and V has no G-stable subspace U with $0 \neq U \neq V$.

For smooth representation (π_i, V_i) of G, the set $\mathrm{Hom}_G(\pi_1, \pi_2)$ is just the space of linear maps $f : V_1 \to V_2$ commuting with the G-actions: $f \circ \pi_1(g) = \pi_2(g) \circ f$ for all $g \in G$. With this definition, the class of smooth representation of G forms a category $\mathrm{Rep}(G)$. We remark that the category Rep (G) is *abelian*.

Question 2.2.1.1. What is abelian category?

We say two smooth representations $(\pi_1, V_1), (\pi_2, V_2)$ of G are *isomorphic*, or *equivalent*, if there exists $\mathbb{C}$-isomorphism $f : V_1 \to V_2$ which is also homomorphism of representations.

Example 2.2.1.2. A character of G can be viewed as representation $\chi : G \to \mathbb{C}^\times = \mathrm{Aut}_{\mathbb{C}}(\mathbb{C})$. The representation $(\chi, \mathbb{C})$ is smooth as $\ker \chi$ is open according to proposition 2.1.6, and since G is locally profinite, $\ker K$ contains a compact open subgroup K of G, implying $\chi(K) = 1_{\mathbb{C}}$. A one-dimensional representation of G is smooth iff it is equivalent to a representation defined by a character of G. It suffices to show that a one-dimensional smooth representation is also a character of G. Indeed, being smooth implies there exists compact open subgroup K of $\ker \chi$. This implies gK also open, implying $\ker \chi$ is open. From proposition 2.1.6, we find χ is continuous, meaning χ is a character. ⌟

example_smoothRep_profinite **Example 2.2.1.3.** Suppose G is compact, hence profinite. Let (π, V) be irreducible smooth representation of G. The space V is then finite-dimensional. For, if $v \in V, v \neq 0$, then $v \in V^K$, for a compact open subgroup K of G. The index $(G : K)$ is finite (properties of compact topological groups) and the set $\{\pi(g)v : g \in G/K\}$ spans V (since V is irreducible, so $\{\pi(g)v : g \in G\}$ spans V, and note $\pi(k)v = v$ for $k \in K$). This implies V is finite-dimensional.

Further, if $K' = \bigcap_{g \in G/K} gKg^{-1}$, then K' is an open (intersection of open sets) normal subgroup (not hard to prove) of finite index (since $[G : H \cap K] \leq [G : H][G : K]$), acting trivially on V (for $v' \in V$ then $v' = \pi(g)v$ for some $g \in G/K$, hence with $k' \in K, k' = gkg^{-1}$, we find $\pi(k')v' = v'$). Thus, as V is irreducible representation of G, we find V an irreducible representation of finite discrete group G/K'. ⌟

2.2.2 Semisimple

section2.2

semisimple_criterion

Proposition 2.2.2.1. Let G be locally profinite group, and let (π, V) be smooth representation of G. The following conditions are equivalent:

1. V is the sum of its irreducible G-subspaces;
2. V is the direct sum of a family of irreducible G-subspaces;
3. any G-subspace of V has G-complement in V.

Proof. We start with (1) $\implies$ (2). Take family $\{U_i : i \in I\}$ of irreducible G-subspaces U_i of V such that $V = \sum_{i\in I} U_i$. We consider the set $\mathcal{I}$ of subsets J of I such that the sum $\sum_{i\in J} U_i$ is direct. The set $\mathcal{I}$ is nonempty, we will show it is inductively ordered by inclusion. For, suppose we have a totally ordered set $\{J_a : a \in A\}$ of elements of $\mathcal{I}$. Put $J = \bigcup_{a\in A} J_a$. If the sum $\sum_{j\in J} U_j$ is not direct, there is a finite subset S of J for which $\sum_{j\in S} U_j$ is not direct. Since S is finite, and $\{J_a : a \in A\}$ is a totally ordered set, there exists $a \in A$ so $S \subset J_a$, which implies a contradiction. Therefore, $J \in \mathcal{I}$. We apply Zorn lemma to get maximal element J_0 of $\mathcal{I}$, for this we have $V = \bigoplus_{i\in J_0} U_i$ as required for (2).

In (3), let W a G-subspace of V. By (2), we can assume $V = \bigoplus_{i\in I} U_i$, for a family (U_i) of irreducible G-subspaces of V. We consider the set $\mathcal{J}$ of subsets J of I for which $W = \cap \sum_{i\in J} U_i = 0$. Again, set $\mathcal{J}$ is nonempty and inductively ordered by inclusion. If J maximal element of $\mathcal{J}$, the sum $X = W + \sum_{j\in J} U_j$ is direct. If $X \neq V$, there is $i \in I$ so $U_i \not\subset X$, so the sum $X + U_i$ is direct, and so $J \cup \{i\} \in \mathcal{J}$, contrary to the hypothesis. Thus (2) $\implies$ (3).

Suppose (3) holds. Let V_0 sum of all irreducible G-subspaces of V and write $V = V_0 \oplus W$ for some G-subspace W of G. Assume for a contradiction that $W \neq 0$. By its definition, space W has no irreducible G-subspace. However, there is a nonzero G-subspace W_1 of W which is finitely generated over G (e.g. take $W_1 = Gvw$ for some $w \in W$). By Zorn's lemma, W_1 has maximal G-subspace W_0, and then W_1/W_0 is irreducible (due to the fact W_0 being maximal). Due to (3), we have $V = V_0 \oplus W_0 \oplus U$ for some G-subspace U of G and hence a G-projection $V \to U$. This projection restricted to W_1 will have kernel W_0 so the image of W_1 in U is isomorphic to W_1/W_0, which is irreducible G-subspace of U. This irreducible subspace is not contained in V_0, a contradiction. Thus, $V = V_0$ and (3) $\implies$ (1). ☐

One says (π, V) is *semisimple* if it satisfies the condition of the above proposition. Interesting locally profinite group G usually have many representations which are not semisimple. However, we have the following:

Lemma 2.2.2.2. Let G locally profinite group, let K compact open subgroup of G. Let (π, V) be smooth representation of G. The space V is the sum of its irreducible K-subspaces.

Proof. View V as representation of K and repeat same argument as in example 2.2.1.3: let $v \in V$, v is fixed by an open normal subgroup K' of K, and it generates a finite-dimensional K-space W on which K' acts trivially (i.e. if v fixed by compact open K'' of K then $W = \{\pi(g)v : g \in K/K''\}$ and $K' = \bigcap_{g \in K/K''} gK''g^{-1}$). W is finite-dimensional representation of finite discrete K/K' and so is the sum of its irreducible K subspaces (Maschke's theorem?). Since $v \in V$ is chosen at random, the lemma follows. □

Question 2.2.2.3. Smooth representation of discrete finite group is the same as normal representaion of that group? If that is the case then we've used Maschke's theorem to above?

The lemma says that V is K-semisimple.

2.2.3 Decompose rep V of G into K-subspaces

section2.3

Let G be a locally profinite group and K compact open subgroup of G. Let $\widehat{K}$ denote the set of equivalence classes of irreducible smooth representations of K. If $\rho \in \widehat{K}$ and (π, V) a smooth representation of G, we define V^ρ to be the sum of all irreducible K-subspaces of V of class ρ. We call V^ρ the *ρ-isotypic component of V*. In particular, V^K is the isotypic subspace for class of trivial representation of K.

Proposition 2.2.3.1. Let (π, V) be a smooth representation of G and let K be compact open subgroup of G.

1. The space V is the direct sum of its K-isotypic components $V = \bigoplus_{\rho \in \widehat{K}} V^\rho$.

2. Let (σ, W) be a smooth representation of G. For any G-homomorphism $f : V \to W$ and $\rho \in \widehat{K}$, we have $f(V^\rho) \subset W^\rho$ and $W^\rho \cap f(V) = f(V^\rho)$.

Proof. Since V is K-semisimple so we write $V = \bigoplus_{i \in I} U_i$ for family of irreducible K-subspaces U_i of V. We let $U(\rho)$ be the sum of those U_i of class ρ. We then have $V = \bigoplus_{\rho \in \widehat{K}} U(\rho)$. Note that U_i is just family of irreducible K-subspaces, not all irreducible K-subspaces. Hence, to show $U(\rho) = V^{\rho)}$ we need to show that for W irreducible K-subspace of V of class ρ then $W \subset U(\rho)$: otherwise there would be a nonzero K-homomorphism $W \to U_i$, for some U_i of class $\tau \neq \rho$.

In (2), image of V^ρ is sum of irreducible K-subspaces of W, all of class ρ and therefore contained in W^ρ. Moreover, $f(V)$ is the sum of images $f(V^\rho), \tau \in \widehat{K}$ and $f(V^\rho) \subset W^\rho$. Since the sum of W^ρ is direct, $f(V)$ is the direct sum of the $f(V^\rho)$ and the second assertion follows. □

We frequently use part (2) of the proposition in following context:

Corollary 2.2.3.2. Let $a : U \to V, b : V \to W$ be G-homomorphisms between smooth representations U, V, W of G. The sequence

$$U \xrightarrow{a} V \xrightarrow{b} W$$

is exact iff

$$U^K \xrightarrow{a} V^K \xrightarrow{b} W^K$$

is exact, for every compact open subgroup K of G.

Question 2.2.3.3. Prove this corrolary.

If H subgroup of G, we define $V(H)$ to be the linear span of $\{v - \pi(h)v : v \in V, h \in H\}$. In particular, $V(H)$ is an H-subspace of V.

Corollary 2.2.3.4. Let G be locally profinite group, and let (π, V) be smooth representation of G. Let K compact open subgroup of G. Then

$$V(K) = \bigoplus_{\rho \in \widehat{K}, \rho \neq 1} V^\rho, V = V^K \oplus V(K),$$

and $V(K)$ is the unique K-complement of V^K in V.

Proof. Sum $W = \bigoplus V^\rho$ with ρ not trivial, is K-complement of V^K in V so there is K-surjection $V \to V^K$ with kernel W. Note $V(K)$ is contained in kernel of any K-homomorphism $V \to V^K$ (indeed, $f(v - \pi(k)v) = f(v) - f(\pi(h)v) = f(v) - \pi(k)f(v) = 0$) so W contains $V(K)$. On the other hand, if U an irreducible K-space of class $\rho \neq 1$, then $V(K) \supset U(K) = U$ implying $V^\rho \subset V(K)$ so $W \subset V(K)$. $\square$

Question 2.2.3.5. Prove that K-complement $V(K)$ of V^K is unique?

AbstractToSmoothRep EXERCISE 5.

1. Let (π, V) be an abstract (not necessarily smooth) representation of G. Define $V^\infty = \bigcup_K V^K$ where K ranges over the compact open subgroups of G. Show that V^∞ is G-stable subspace of V. Define homomorphism $\pi^\infty : G \to \mathrm{Aut}_{\mathbb{C}}(V^\infty)$ by $\pi^\infty(g) = \pi(g)|_{V^\infty}$. Show that (π^∞, V^∞) is a smooth representation of G.
2. Let (π, V) be smooth representation of G and (σ, W) an abstract representation. Let $f : V \to W$ be G-homomorphism. Show that $f(V) \subset W^\infty$, hence, $\mathrm{Hom}_G(V, W) = \mathrm{Hom}_G(V, W^\infty)$.
3. Functor $V \mapsto V^\infty$ is left-exact.

2.2.4 Smooth induction

section2.4

Let G locally profinite group, and let H closed subgroup of G. Thus H is also locally profinite.

Let (σ, W) smooth representation of H. We consider the space X of functions $f : G \to W$ which satisfy

1. $f(hg) = \sigma(h)f(g), h \in H, g \in G;$

2. there is a compact open subgroup K of G (depending on f) such that $f(gx) = f(g)$ for $g \in G, x \in K$.

We define homomorphism $\sum : G \to \mathrm{Aut}_{\mathbb{C}}(X)$ by $\sum(g)f : x \mapsto f(xg)$ for $x, g \in G$. The pair $(\sum, X)$ provides a smooth representation of G [1]. It is called the representation of G *smoothly induced by σ*, and is usually denoted $(\sum, X) = \mathrm{Ind}_H^G \sigma$. The map $\sigma \mapsto \mathrm{Ind}_H^G \sigma$ gives a functor $\mathrm{Rep}(H) \to \mathrm{Rep}(G)$.

Remark 2.2.4.1. Function satisfying only the second condition is said to be *right locally constant*. Two notions of left and right locally constant are actually coincides, since $gx = x(x^{-1}gx)$.

There is a canonical H-homomorphism

$$\alpha_\sigma : \mathrm{Ind}_H^G \sigma \to W, f \mapsto f(1).$$

The pair $(\mathrm{Ind}_H^G \sigma, \alpha)$ has following fundamental property:

Theorem 2.2.4.2 (Frobenius reciprocity)**.** Let H closed subgroup of locally profinite group G. For a smooth representation (σ, W) of H and a smooth representation (π, V) of G, the canonical map

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \sigma) \to \mathrm{Hom}_H(\pi, \sigma), \phi \mapsto \alpha_\sigma \circ \phi,$$

is an isomorphism that is functorial in both variables π, σ.

Proof. Let $f : V \to W$ be H-homomorphism. We define G-homomorphism $f_\star : V \to \mathrm{Ind}_H^G \sigma$ by sending v to $g \mapsto f(\pi(g)v)$. The map $f \mapsto f_\star$ is inverse of canonical map mentioned in the theorem. Indeed, the map in the theorem will send $f_\star$ to H-homomorphism $a_\sigma \circ f_\star : V \to \mathrm{Ind}_H^G \sigma \to W$ which sends

$$v \mapsto (g \mapsto f(\pi(g)v)) \mapsto f(\pi(1)v) = f(v).$$

□

Question 2.2.4.3. What does functoriality in π, σ mean? Addition?

A simple consequence is that $a_\sigma(V) \neq 0$, for any nonzero G-subspace V of $\mathrm{Ind}_H^G \sigma$.

Proof. First, $\mathrm{Ind}_H^G : \mathrm{Rep}(H) \to \mathrm{Rep}(G)$ is a functor. In particular, the morphisms between two categories via this functor is as follows: If $f : (\sigma, V) \to (\tau, W)$ a H-homomorphism. Under Ind_H^G, σ, τ are sent to $X(\sigma), X(\tau)$. We define $\mathrm{Ind}_H^G f : X(\sigma) \to X(\tau)$ as for $f' \in X(\sigma)$, $f' \mapsto f \circ f'$.

For smooth representation (σ, W) of H, temporarily let $I(\sigma)$ denote the space of functions $G \to W$ satisfying the first condition $f(gh) = \sigma(h)f(g)$ of the definition above. Thus I is a functor to the category of abstract representation of G, sending (σ, W) to $(\sum, I(\sigma))$. We check that I is exact and additive:

[1] it is smooth because for any $f \in X$, there is compact open subgroup K of G such that $f(gx) = f(g)$ for $g \in G, x \in K$, implying $\sum(g)f = f$ for all $g \in K$

1. Note $\mathrm{Rep}(G)$ is preadditive as $\mathrm{Hom}(A, B)$ where A, B (abstract or smooth) representation of G is abelian group under usual addition in vector spaces A, B.

2. For $(\sigma, A), (\tau, B)$ representation of H. Under I, these two are sent to $(\sum_\sigma, I(\sigma))$ and $(\sum_\tau, I(\tau))$. We need to show $I : \mathrm{Hom}(A, B) \to \mathrm{Hom}(I(\sigma), I(\tau))$ is group homomorphism. Recall I sends $f \in \mathrm{Hom}(A, B)$ to $I(f) : f' \mapsto f \circ f'$. For $f, g \in \mathrm{Hom}(A, B)$, we have $I(f + g) : (f' : G \to A) \mapsto (f + g) \circ f'$ and since $(f + g) \circ f' = f \circ f' + g \circ f'$ so $I(f + g) = I(f) + I(g)$, as desired. Thus, I is additive.

3. ... I guess it's not hard to show I is also exact based on the definition. Later ...

We find $\mathrm{Ind}_H^G(\sigma) = I(\sigma)^\infty$. It's not hard to show $\mathrm{Ind}_H^G(\sigma)$ is additive and exercise 5 shows it is left-exact.

To prove it is right-exact, let $(\sigma, W), (\tau, U)$ be smooth representations of H and let $f : W \to U$ be H-surjection, i.e. $W \to U \to 0$ an exact sequence. We want to show $I(\sigma)^\infty \to I(\tau)^\infty \to 0$ is an exact sequence.

Take $\phi \in I(\tau)^\infty$ and choose compact open subgroup K of G which fixes ϕ, i.e. $\phi(x) = \phi(xK)$ for all $x \in G$. The support of $\phi : G \to U$ is the union of cosets HgK (so $\phi(hgk) = \tau(h)\phi(g)$ and as $\tau : H \to \mathrm{Aut}_{\mathbb{C}}(U)$ we have $\phi(hgk) \neq 0$ as long as $\phi(g) \neq 0$), and $\phi(g) \in U$ must be fixed by $\tau(H \cap gKg^{-1})$ (since $H \cap gKg^{-1} \in H$ so $\tau(H \cap gKg^{-1})\phi(g) = \phi((H \cap gKg^{-1})g)$ and recall $\phi(x) = \phi(xK)$). By 2.2.3 corollary 1, applied to exact sequence $W \to U \to 0$ (since $f : W \to U$ is surjective) as representations of H and compact open subgroup $gKg^{-1} \cap H$ of H (it is closed as intersection of closed subgroups and is in compact gKg^{-1} so it is compact; gKg^{-1} is open relative to H?), we obtain exact sequence $W^{gKg^{-1} \cap H} \to U^{gK^{-1} \cap H} \to 0$. As $\phi(g) \in U^{gK^{-1}g \cap H}$, there exists $w_g \in W$, fixed by $\sigma(gKg^{-1} \cap H)$, such that $f(w_g) = \phi(g)$. We define a function $\Phi : G \to W$ to have the same support as ϕ and $\Phi(hgk) = \sigma(h)w_g$, for each $g \in H \backslash \mathrm{supp}\phi / K$. This function is fixed by K, i.e. $\sum(k)\Phi = \Phi \iff \Phi(xk) = \Phi(x)$ and hence lies in $I(\sigma)^\infty$. Its image is ϕ, as required. $\square$

Question 2.2.4.4. Why proving right-exactness only to $W \to U \to 0$ but not $V \to W \to U \to 0$?

2.2.5 Smooth induction with compact supports

With (σ, W) and X as in 2.2.4, consider the space X_c of functions $f \in X$ which are *compactly supported modulo H*: this means that the image of support $\mathrm{supp} f \subset G$ [2] of f in $H \backslash G$ is compact or, equivalently, $\mathrm{supp} f \subset HC$, for some compact set C

[2] As $f \in X$ which is left K-invariant for some open K, set of $x \in G$ so $f(x) \neq 0$ is union of open sets xK, hence open, hence closed. This means we don't have to take closure to obtain supp f in this case

in G [3]. The space X_c is stable under action of G and provides another smooth representation of G. It is denoted $c\text{-Ind}_H^G\sigma$, and gives a functor

$$c\text{Ind}_H^G : \text{Rep}(H) \to \text{Rep}(G).$$

One calls it *compact induction*, or *smooth induction with compact supports*.

Exercise 6. Check X_c provides smooth representation of G.

Exercise 7. Show that the functor $c\text{-Ind}_H^G$ is additive and exact.

In all cases, there is a canonical G-embedding $c\text{-Ind}_H^G\sigma \to \text{Ind}_H^G\sigma$, or morphism of functors $c\text{-Ind}_H^G \to \text{Ind}_H^G\sigma$. This is an isomorphism if and only if $H\backslash G$ is compact [4]. On the other hand, for specific H, G, σ, the map $c\text{-Ind}_H^G\sigma \to \text{Ind}_H^G\sigma$ can be isomorphism even when $H\backslash G$ is not compact, which will be considered in later section.

This construction is mainly of interest when the subgroup H is *open* in G: In this case, there is a canonical H-homomorphism

$$\alpha_\sigma^c : W \to c\text{Ind}\sigma, w \mapsto f_w.$$

where $f_w \in X_c$ is supported in H and $f_w(h) = \sigma(h)w, h \in H$.

Exercise 8. Suppose H is open in G. Let $\phi : G \to W$ be a function, compactly supported modulo H, such that $\phi(hg) = \sigma(h)\phi(g), h \in H, g \in G$. Show that $\phi \in X_c$.

To show this, it suffices to prove that there is a compact open subgroup K of G such that $\phi(gx) = \phi(g)$ for all $g \in G, x \in K$.

ϕ is compactly supported modulo H implies $\text{supp}(\phi) = \overline{\{g \in G : \phi(g) \neq 0\}}$ is compact in $H\backslash G$. Since H is open, $H\backslash G$ is a discrete topology, and as $\phi(hx) = \sigma(h)\phi(x) \neq 0 \iff \phi(x) \neq 0$, we know $\{c \in H\backslash G, \phi(c) \neq 0\}$ is an open (disjoint) cover of image of $\text{supp}(\phi)$ in $H\backslash G$. Hence, from our condition, we know that $\{c \in H\backslash G, \phi(c) \neq 0\}$ is finite, i.e. we can write this as $\{c_i : 1 \le i \le n, c_i \in G\}$. As $G \to H\backslash G$ is an open map, we know $\{Hc_i : 1 \le i \le n, c_i \in G\}$ is an open subcover of $\text{supp}(\phi)$. Since W is a smooth representation of H, for each $\phi(c_i) \in W$, there exists compact open subgroup K_i of H such that $\sigma(k)\phi(c_i) = \phi(c_i)$ for all $k \in K_i$. Take $K = \bigcap K_i$, we obtain that K is nonempty compact open subgroup (as $1 \in K$) such that $\phi(kc_i) = \phi(c_i)$ for all $1 \le i \le n, k \in K$. Let $\widehat{K} = \bigcap c_i^{-1}Kc_i$ is also a nonempty compact open subgroup such that $\phi(c_i\widehat{k}) = \phi(c_i)$ (so here we switch from left to right locally constant).

We show that $\widehat{K}$ is our desired compact open subgroup. For any $g \in G, \phi(g) \neq 0$ then $g = hc_i$ for some $h \in H, 1 \le i \le n$. Hence, for any $k \in \widehat{K}$, we have $\phi(gx) = \sigma(h)\phi(c_ik) = \sigma(h)\phi(c_i) = \phi(g)$. On the other hand, if $\phi(g) = 0$ then we show $\phi(gx) = 0$ for all $x \in \widehat{K}$. Indeed, if not, then $gx = hc_i$ and as $x \in \widehat{K} \subset c_i^{-1}Kc_i$ so exists $k \in K$ so $g(c_i^{-1}kc_i) = hc_i$ implying $g = hk^{-1}c_i$. But that means $\phi(g) = \sigma(h)\phi(k^{-1}c_i)\sigma(h)\phi(c_i) \neq 0$, a contradiction. Thus, $\phi(gx) = 0 = \phi(g)$.

[3] If image of supp f in $H\backslash G$ has representative sets $C \subset G$ then supp $f \subset HC$. Consider open cover $\{B_i\}$ of C in G, then $\{HB_i\}$ covers $\{Hc : c \in C\}$ in $H\backslash G$, implying $\{Hc : c \in C\}$ has finite cover in $H\backslash G$ from the definition, and then going backward to G via π^{-1} to get that C has finite open cover

[4] If $H\backslash G$ is compact, as supp f is open since $f \in X$ in section 2.2.4, open $\pi(\text{supp } f) \subset H\backslash G$ is open, hence closed, hence compact, hence the condition of being compactly supported modulo H is vacuous. Conversely, if the embedding is an isomorphism, meaning space $X = X_c$, i.e. every $f \in X$ is compactly supported modulo H show the converse

Lemma 2.2.5.1. Let H be open subgroup of G, and let (σ, W) be smooth representation of H.

1. The map $\alpha_\sigma^c : w \mapsto f_w$ is an H-isomorphism of W with the space of functions $f \in c\text{-Ind}_H^G\sigma$ such that $\text{supp} f \subset H$.
2. Let $\mathcal{W}$ be $\mathbb{C}$-basis of W, and $\mathcal{G}$ a set of representations for G/H. The set $\{gf_w : w \in \mathcal{W}, g \in \mathcal{G}\}$ is a $\mathbb{C}$-basis of $c\text{-Ind}_H^G\sigma$.

Proof. α_σ^c is an H-homomorphism since we have

$$\sum(h)f_w(x) = f_w(xh) = \begin{cases} 0 & x \notin H, \\ \sigma(xh)w \end{cases} = f_{\sigma(h)w}(x).$$

It has the inverse map $f \mapsto f(1)$ as $\alpha_\sigma^c(f(1))(h) = \sigma(h)f(1) = f(h)$.

As argue similarly in the exercise, the support of function $f \in c\text{-Ind}_H^G\sigma$ is finite union of cosets Hg^{-1}, for $g \in \mathcal{G}$, and the restriction of f to any one of these cosets also lies in c-Indσ. With this then f is written as finite sum of these restrictions. If $\text{supp} f = Hg^{-1}$ then $g^{-1}f$ has support contained in H so from part (1), it is a finite linear combination of $f_w, w \in \mathcal{W}$. This follows such f is finite linear combination of $gf_w, w \in \mathcal{W}$. Such set of functions $gf_w, w \in \mathcal{W}, g \in \mathcal{G}$ is linearly independent (may need to elaborate this, but later), so we are done. □

For open subgroups, compact induction has its own form of Frobenius Reciprocity:

Proposition 2.2.5.2. Let H open subgroup of G, let (σ, W) be a smooth representation of H and (π, V) a smooth representation of G. The canonical map

$$\text{Hom}_G(c\text{-Ind}\sigma, \pi) \to \text{Hom}_H(\sigma, \pi|_H), f \mapsto f \circ \alpha_\sigma^c$$

is an H-isomorphism that is functorial in both variables.

Proof. Not hard to figure out the inverse map. □

2.2.6 Schur's lemma

section2.6

It is convenient to introduce technical restriction o group G. *From now on, we assume that:*

For any compact open subgroup K of G, set G/K is countable.

Remark if G/K is countable for one compact open subgroup K of G, then G/K' is countable for any compact open subgroup K' of G. For, $K \cap K'$ is compact, open, and of finite index in K (Proposition 1.2.2.2). Thus, the surjection $G/(K \cap K') \to G/K$ has finite fibres, and since G/K is countable, so is $G/(K \cap K')$. Then surjection $G/(K \cap K') \to G/K'$ also has finite fibres, implying G/K' is countable.

The main effect of the above assumtion is

Lemma 2.2.6.1. Let (π, V) irreducible smooth representation of G. The dimension $\dim_{\mathbb{C}} V$ is countable.

Proof. Let $v \in V, v \neq 0$ and choose compact open subgroup K of G such that $v \in V^K$. Since V is irreducible, the countable set $\{\pi(g)v : g \in G/K\}$ spans V. □

This enables us to generalize familiar result:

Lemma 2.2.6.2 (Schur)**.** If (π, V) irreducible smooth representation of G, then $\mathrm{End}_G(V) = \mathbb{C}$.

Proof. Let $\phi \in \mathrm{End}_G(V), \phi \neq 0$. The image and kernel of ϕ are G-subspaces of V, so ϕ is bijective and invertible. Therefore, $\mathrm{End}_G(V)$ is complex division algebra.

If we fix $v \in V, v \neq 0$, the G-translates of v span V so an element $\phi \in \mathrm{End}_G(V)$ is determined uniquely by $\phi(v) \in V$. Since $\dim_{\mathbb{C}} V$ is countable, we deduce that $\mathrm{End}_G(V)$ has countable dimension (under addition $+$).

For any $\phi \in \mathrm{End}_G(V), \phi \neq \mathbb{C}$, is transcendental over $\mathbb{C}$. Indeed, if we have $\sum_{i=0}^n a_i \phi^i = 0$ where ϕ^i denotes composition i times, since it's over $\mathbb{C}$, one obtain linear factorization $\sum_{i=1}^n (\phi - b_i) = 0$. Take any $v \in V, v \neq 0$ and apply to this linear map, one finds that at some point, there must exists some i such that $(\phi - b_i)w = 0$ for nonzero $w \in V$. Since $\{gw : w \in V\}$ spans V and note $\phi(gw) = g\phi(w) = b_i(gw)$ so $\phi = \mathbb{C}$, a contradiction.

Thus, ϕ is transcendental over $\mathbb{C}$, which generates field $\mathbb{C}(\phi) \subset \mathrm{End}_G(V)$. The subset $\{(\phi - a)^{-1} : a \in \mathbb{C}\}$ of $\mathbb{C}(\phi)$ is linearly independent over $\mathbb{C}$ (why?), so the $\mathbb{C}$-dimension of $\mathbb{C}(\phi)$ is uncountable, and this is impossible. Thus, we conclude $\mathrm{End}_G(V) = \mathbb{C}$. □

Corollary 2.2.6.3. Let (π, V) be irreducible smooth representation of G. The centre Z of G acts on V via a character $\omega_\pi : Z \to \mathbb{C}^\times$, that is, $\pi(z)v = \omega_\pi(z)v$ for $v \in V$ and $z \in Z$.

Proof. By Schur's lemma, since V is irreducible, there is a homomorphism $\omega_\pi : Z \to \mathbb{C}^\times$ such that $\pi(z)v = \omega_\pi(z)v, z \in Z, v \in V$. If K is compact open subgroup of G such that $V^K \neq 0$, then there exists $v \in V, v \neq 0$ such that $\pi(z)v = v$ for $z \in Z \cap K$, implying $\omega_\pi(z) = 1$ for all $z \in Z \cap K$. Thus ω_π is trivial on the compact open subgroup $K \cap Z$ of Z, implying ω_π is continuous or ω_π is a character of Z. □

One calls ω_π the *central character* of π.

Corollary 2.2.6.4. If G is abelian, any irreducible smooth representation of G is one-dimensional.

Remark 2.2.6.5. If G is compact, the converse of Schur's lemma holds: a smooth representation (π, V) of G is a direct sum of irreducible representations, so $\mathrm{End}_G(V)$ is one-dimensional iff π is irreducible.

2.2.7 G-semisimple iff H-semisimple for open subgroup H of finite index

section2.7

Lemma 2.2.7.1. Let G be locally profinite group, and let H open subgroup of G of finite index.

1. If (π, V) is a smooth representation of G, then V is G-semisimple if and only if V is H-semisimple.
2. Let (σ, W) be semisimple smooth representation of H. The induced representation $\mathrm{Ind}_H^G \sigma$ is G-semisimple.

Proof. Suppose that V is semisimple, and let U be G-subspace of V. By hypothesis, there is H-subspace W of V such that $V = U \oplus W$. Let $f : V \to U$ be the H-projection with kernel W. Consider the map

$$f^G : v \mapsto (G:H)^{-1} \sum_{g \in G/H} \pi(g) f(\pi(g)^{-1} v), v \in V.$$

This definition is independent of the choice of cosets representatives and it follows that f^G is a G-projection $V \to U$. Indeed, we check those:

1. It is independent of choice of cosets representatives, as

$$\pi(gh)f(\pi(gh)^{-1}v) = \pi(g)\pi(h)f(\pi(h)^{-1}\pi(g)^{-1}v) = \pi(g)f(\pi(g)^{-1}v)$$

   as f is H-homomorphism.

2. It is G-projection. Firstly, it is G-homomorphism as

$$\begin{aligned}\pi(g')f^G(v) &= (G:H)^{-1} \sum_{g \in G/H} \pi(g'g) f(\pi(g)^{-1}v), v \in V, \\ &= (G:H)^{-1} \sum_{g \in G/H} \pi(g) f(\pi(g'^{-1}g)^{-1}v), \\ &= f^G(\pi(g')v).\end{aligned}$$

   Why f^G projection? It seems I couldn't show $(f^G)^2 = f^G$?

We then have $V = U \oplus \ker f^G$ and $\ker f^G$ is a G-subspace of V. Thus V is semisimple (2.2.2).

Conversely, suppose V is G-semisimple. Thus G is direct sum irreducible G-subspace (2.2.2), and it is enough to treat case where V is irreducible over G. As representation of H, the space V is finitely generated (V is irreducible over G so $\{\pi(g)v : g \in G\}$ spans V, and since G/H is finite so V H-finitely generated by $\{\pi(g)v : g \in G/H\}$) so it admits irreducible H-quotients U (i.e. H-subspace of V that is irreducible over H). Suppose for the moment that H is *normal* subgroup of

G. By Frobenius Reciprocity 2.2.4, the H-map $V \to U$ gives a nontrivial, hence injective (since V is G-irreducible) G-map $V \hookrightarrow \mathrm{Ind}_H^G U$. As representation of H, the induced representation $\mathrm{Ind}_H^G U = c\text{-}\mathrm{Ind}_H^G U$ (since G/H is finite) is a direct sum of G-conjugates of U (from lemma in 2.2.5, U isomorphic to space $\mathcal{C}$ of functions supported in H, and then as G/H begin finite, $c\text{-}\mathrm{Ind}_H^G U$ is direct sum of $g\mathcal{C}, g \in G/H$ which is spanned by $\{gf_w : w \in \mathcal{W}\}$). These are all irreducible over H (first, it is indeed a representation of H isomorphic to U and then U is irreducible) so $\mathrm{Ind}U$ is H-semisimple. Proposition 2.2.2.1 then implies $V \subset \mathrm{Ind}U$ is H-semisimple.

In general, we set $H_0 = \bigcap_{g \in G/H} gHg^{-1}$ which gives us open compact normal subgroup of G of finite index. We have just shown that G-space V is H_0-semisimple. The first part of the proof (turning a H_0-projection into H-projection) shows V is H-semisimple. □

We apply the lemma in following context. Let Z center of G, and fix character χ of Z. Consider class of smooth representations (π, V) of G which *admits χ as central character*, that is,

$$\pi(z)v = \chi(z)v, z \in Z, v \in V.$$

Proposition 2.2.7.2. Let (π, V) be smooth representation of G, admitting χ as central character. Let K open subgroup of G such that KZ/Z is compact.

1. Let $v \in V$. The KZ-space spanned by v is of finite dimension, and is sum of irreducible KZ-spaces.

2. As representation of KZ, the space V is semisimple.

Proof. The vector $v \in V$ is fixed by a compact open subgroup K_0 of K (indeed, say v fixed by compact open $K' \subset G$ as G is locally profinite so v also fixed by $K_0 = K' \cap K$ which is compact (as it is closed in K') open subgroup of K). KZ/K_0Z is finite as K_0 is compact of K, so the space W spanned by $\pi(KZ)v$ has finite dimension. Indeed, note $\pi(k_0z)v = \chi(z)v$ so $\pi(K_0Z)v$ is 1-dimensional), as desired. This also implies W is K_0Z-semisimple. The lemma above implies W is KZ-semisimple. Since v was chosen arbitrarily, the second assertion follows. □

In practice, open subgroup K will contain Z, with K/Z compact. The discussion is equally valid if Z closed subgroup of the center $Z(G)$ of G.

2.2.8 Contragedient/Smooth dual

section2.8 S

Let (π, V) smooth representation of locally profinite group G. Write $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$, and denote by

$$V^* \times V \to \mathbb{C}, (v^*, v) \mapsto \langle v^*, v \rangle = v^*(v),$$

the canonical evaluation pairing. The space V^* carries a representation π^* of G defined by

$$\langle \pi^*(g)v^*, v\rangle = \langle v^*, \pi(g^{-1})v\rangle, g \in G, v^* \in V^*, v \in V.$$

That is not, in general, smooth. We accordingly define

$$\check{V} = (V^*)^\infty = \bigcup_K (V^*)^K,$$

where K ranges over compact open subgroup of G. exercise 5 shows $\check{V}$ is G-table subspace of V^*, and provides smooth representation

$$\check{\pi} = (\pi^*)^\infty : G \to \mathrm{Aut}_{\mathbb{C}}(\check{V}).$$

The representation $(\check{\pi}, \check{V})$ is called the *contragedient*, or *smooth dual*, of (π, V). We continue to denote the evaluation pairing $\check{V} \times V \to \mathbb{C}$ by $(\check{v}, v) \mapsto \langle \check{v}, v\rangle$. Therefore,

$$\langle \check{\pi}(g)(\check{v}), v\rangle = \langle \check{v}, \pi(g^{-1})v\rangle, \check{v} \in \check{V}, g \in G, v \in V. \tag{2.2.8.1}$$

{contragedient_evaluation}

Let K be compact open subgroup of G. We recall V^K has unique K-complement $V(K)$ in V (2.2.3). If $\check{v} \in \check{V}$ is fixed under$K$, we have $\langle \check{v}, V(K)\rangle = 0$. Indeed, we have $\check{\pi}(g)\check{v} = \check{v}$ for $g \in K$, implying

$$\langle \check{\pi}(g)\check{v}, v\rangle = \langle \check{v}, v\rangle \iff \langle \check{v}, \pi(g^{-1})(v) - v\rangle = 0, g \in K, v \in V.$$

As $g \in K$, we know $\pi(g)v - v = 0$ when $v \in V^K$. Hence, $\pi(g)v - v \in V(K)$ for $v \in V$. This type of elements certainly span $V(K)$ (need to check this, note V is not necessarily finite dimensional). In the end, we obtain $\langle \check{v}, V(K)\rangle = 0$. Thus, $\check{v} \in \check{V}^K$ is determined by its effect on V^K.

Proposition 2.2.8.1. Restriction to V^K induces an isomorphism $\check{V}^K \cong (V^K)^*$.

Proof. We already obtained a map $\check{V}^K \to (V^K)^*$. For the inverse map, one can extend a linear functional on V^K to an element of $\check{V}^K$ by letting it be trivial on $V(K)$. □

Corollary 2.2.8.2. Let (π, V) be smooth representation of G, and let $v \in V, v \neq 0$. There exists $\check{v} \in \check{V}$ such that $\langle \check{v}, v\rangle \neq 0$.

Proof. Since (π, V) is smooth, for any $v \in V, v \neq 0$, there exists open compact subgroup K so $v \in V^K$. One can define a linear functional f on V^K that is nonzero at v. By the proof of above proposition, f can be extended to an element $\check{v}$ of $\check{V}^K$. □

Remark 2.2.8.3. Subspace $\check{V}$ of V^* does depend on G in the following sense: Let H closed subgroup of G, and let $\tilde{V}$ space of H-smooth vectors in V^*. Then certainly $\check{V} \subset \tilde{V}$, but there can be situation where $\tilde{V} \neq \check{V}$.

2.2.9 Isomorphism with double smooth dual iff admissible

section2.9

Let (π, V) smooth representation of locally profinite group G. We can form the smooth dual $(\check{\check{\pi}}, \check{\check{V}})$ of $(\check{\pi}, \check{V})$. There is a G-canonical map $\delta : V \to \check{\check{V}}$ by

$$\langle \delta(v), \check{v} \rangle_{\check{V}} = \langle \check{v}, v \rangle_V, v \in V, \check{v} \in \check{V}.$$

It is injective (from corollary 2.2.8).

Proposition 2.2.9.1. Let (π, V) smooth representation of locally profinite group G. The canonical map $\delta : V \to \check{\check{V}}$ is an isomorphism if and only if π is admissible.

Proof. The map δ induces a map $\delta^K : V^K \to \check{\check{V}}^K$ for each open compact subgroup K of G. Indeed, for $v \in V^K$, we need to show $\delta(v) \in \check{\check{V}}^K$, which holds since

$$\begin{aligned} \langle \check{\check{\pi}}(g)\delta(v), \check{v} \rangle_{\check{V}} &= \langle \delta(v), \check{\pi}(g^{-1})\check{v} \rangle_{\check{V}}, \\ &= \langle \check{\pi}(g)\check{v}, v \rangle_V, \\ &= \langle \check{v}, \pi(g)v \rangle_V, \\ &= \langle \check{v}, v \rangle_V, \\ &= \langle \delta(v), \check{v} \rangle_{\check{V}} \end{aligned}$$

for all $\check{v} \in \check{V}, g \in K$. Thus, δ is surjective iff δ^K is surjective for all K (exactness property in corollary in 2.2.3). From proposition in 2.2.9, we know δ^K is the canonical map $V^K \to (V^K)^{**}$, which is surjective iff $\dim_{\mathbb{C}} V^K < \infty$. □

2.2.10 Admissible rep V irreducible iff $\check{V}$ is

section2.10

Let $(\pi, V), (\sigma, W)$ be smooth representations of G, and let $f : V \to W$ be a G-map. We define a map $\check{f} : \check{W} \to \check{V}$ by the relation

$$\langle \check{f}(\check{w}), v \rangle_W = \langle \check{w}, f(v) \rangle, \check{w} \in \check{W}, v \in V.$$

The map $\check{f}$ is a G-homomorphism, and $(\pi, V) \to (\check{\pi}, \check{V})$ gives a contravariant functor of $\mathrm{Rep}(G)$ itself.

Lemma 2.2.10.1. The contravariant functor $\mathrm{Rep}(G) \to \mathrm{Rep}(G)$ sending (π, V) to $(\check{g}, \check{V})$ is exact.

Proof. If we have an exact sequence of smooth representations (π_i, V_i) of G:

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$$

the sequence

$$0 \longrightarrow V_1^K \longrightarrow V_2^K \longrightarrow V_3^K \longrightarrow 0$$

is exact according to 2.2.3. The sequence of dual spaces $(V_i^K)^*$ is then exact (nothing about smoothness is involved here), and note $(V_i^K)^* \cong \check{V}_i^K$ and the result follows from 2.2.3. $\square$

We deduce:

Proposition 2.2.10.2. Let (π, V) be admissible representation of G. Then (π, V) is irreducible if and only if $(\check{\pi}, \check{V})$ is irreducible.

Proof. If (π, V) has irreducible subspace U then we have projection $V \to U$ so $V = U \oplus W$ where W is G-space of V. We obtain exact sequence $0 \to U \to V \to W \to 0$. From above lemma, we find $0 \to \check{U} \to \check{V} \to \check{W} \to 0$ also exact sequence, implying $\check{V}$ is reducible.

Conversely, if $(\check{\pi}, \check{V})$ not irreducible then so is $(\check{\check{\pi}}, \check{\check{V}}) \cong (\pi, \check{V})$. $\square$

Exercise 9. Let (π, V) and (σ, W) be smooth representation of G. Let $\mathfrak{P}(\pi, \sigma)$ be space of G-invariant bilinear pairing $V \times W \to \mathbb{C}$. Show that there are canonical isomorphism

$$\mathrm{Hom}_G(\pi, \check{\sigma}) \cong \mathfrak{P}(\pi, \sigma) \cong \mathrm{Hom}_G(\sigma, \check{\pi}).$$

2.3 Measures and Duality

2.3.1 Haar integral on $C_c^\infty(G)$

section3.1

Let G be locally profinite group. Let $C_c^\infty(G)$ space of functions $f : G \to \mathbb{C}$ which are locally constant and of compact support.

Let $f \in C_c^\infty(G)$. Local constancy and compactness of support together imply that there exist compact open subgroup K_1, K_2 of G such that $f(k_1 g) = f(g) = f(gk_2)$, for all $g \in G, k_i \in K_i$. Taking $K = K_1 \cap K_2$ one sees that f is a finite linear combination of characteristic functions of double cosets KgK.

Explanation. The proof for this mimics the one in section 2.2.5. Here *locally constant* $f \in C_c^\infty(G)$ means for every $g \in G$, there exists compact open subgroup $U_g \subset G$ such that $f(gU_g) = f(g)$. *Support* of f on topological space G with values on some vector space is the closure of the set $\{x \in G : f(x) \neq 0\}$. For locally constant function f on locally profinite group G, this set is closed.

Locally constant functions with compact support from G to vector space S are smooth functions.

The *characteristic function* of KgK means function $\iota_{KgK} : G \to \mathbb{C}$ that is 1 on KgK and 0 everywhere else. $\square$

The group G acts on $C_c^\infty(G)$ be *left translation* λ and by *right translation* ρ:

$$\lambda_g f : x \mapsto f(g^{-1}x), \rho_g f : x \mapsto f(xg); x, g \in G, f \in C_c^\infty. \tag{2.3.1.1}$$ {measure_translation}

Both of the G-representations $(C_c^\infty(G), \lambda), (C_c^\infty(G), \rho)$ are smooth.

Definition 2.3.1.1. A *right Haar integral* on G is nonzero linear functional $I : C_c^\infty \to \mathbb{C}$ such that

1. $I(\rho_g f) = I(f), g \in G, f \in C_c^\infty(G)$,
2. $I(f) \geq 0$ for any $f \in C_c^\infty, f \geq 0$.

One defines *left Haar integral* similarly, using left translation λ instead of right translation. We now show that G prossesses only one right Haar integral.

Proposition 2.3.1.2. There exists a right Haar integral $I : C_c^\infty \to \mathbb{C}$. Moreover, a linear functional $I' : C_c^\infty \to \mathbb{C}$ is a right Haar integral iff $I' = cI$, for some constant $c > 0$.

Proof. Let K compact open subgroup of G. We denote ${}^K C_c^\infty(G)$ the space of functions in $C_c^\infty(G)$ fixed by $\lambda(K)$. We view ${}^K C_c^\infty(G)$ as G-space via right translation.It is then identical to the representation of G compactly induced from the trivial representation 1_K of K: ${}^K C_c^\infty(G) = c\text{-Ind}_K^G 1_K$. We check this first:

1. We need to show for $f \in^K C_c^\infty(G)$ then $\rho_g(f) \in^K C_c^\infty(G)$, i.e. $\lambda_k(\rho_g f) = \rho_g f$ for all $k \in K$. Indeed, we have
$$\lambda_k(\rho_g f) : x \mapsto (\rho_g f)(k^{-1}x) = f(k^{-1}xg)$$
Since $f \in^K C_c^\infty(G)$ so $\lambda(K)f = f$, meaning $f(k^{-1}xg) = f(xg) = (\rho_g f)(x)$, as desired.

2. We need to show $\rho : G \to \text{Aut}_{\mathbb{C}}({}^K C_c^\infty(G))$ is a smooth group homomorphism. It is group homomorphism as
$$\rho_{g_1 g_2}(f) : x \mapsto f(xg_1g_2) = \rho_{g_2}(f)(xg_1) = (\rho_{g_1}(\rho_{g_2} f))(x).$$
Since $(C_c^\infty(G), \rho)$ is smooth so $(\rho,^K C_c^\infty(G))$ is also smooth.

3. We show ${}^K C_c^\infty(G) = c\text{-Ind}_K^G 1_K$ by assuming that $\mathbb{C}$ is 1_K. The action of G on these two sets are defined the same. Suppose $f \in c\text{-Ind}_K^G 1_K$ so following the definition, $f(kg) = f(g)$ for all $k \in K, g \in G$, meaning f is fixed under $\lambda(K)$. f is locally constant according to definition. f is compactly supported modulo open compact K, i.e. $\text{supp} f \subset KC$ for compact C of G, and since KC is compact, $\text{supp} f$ is also compact. Thus, $f \in^K C_c^\infty(G)$.

   Conversely, if $f \in^K C_c^\infty(G)$ then the fact that f fixed under $\lambda(K)$ and f locally constant implies $f \in \text{Ind}_K^G 1_K$. To show $f \in c\text{-Ind}_K^G 1_K$, we need f to be compact support modulo K. Since supp f is compact on G so supp f compact on G/K via continuous map $G \to G/K$, as desired.

Lemma 2.3.1.3. Viewing $\mathbb{C}$ as the trivial G-space, we have
$$\dim_{\mathbb{C}} \text{Hom}_G({}^K C_c^\infty(G), \mathbb{C}) = 1.$$
There exists nonzero element $I_K \in \text{Hom}_G({}^K C_c^\infty(G), \mathbb{C})$ such that $I_K(f) \geq 0$ whenever $f \geq 0$. If h_K is the characteristic function of K, then $I_K(h_K) > 0$.

Proof of lemma. The first assertion is given by proposition in section 2.2.5, saying that $\text{Hom}_G({}^K C_c^\infty(G), \mathbb{C}) \cong \text{Hom}_K(1_K, \mathbb{C})$.

For $g \in G$, let f_g denote the characteristic function of Kg then the set of functions $f_g, K\backslash G$, then forms a $\mathbb{C}$-basis of space ${}^K C_c^\infty(G)$ (lemma from section 2.2.5 with the note that $1_K = \mathbb{C}$). The functional $I_K : f_g \to 1$ has the required properties, noting that $h_K = f_1$. □

We choose a descending sequence $\{K_n\}_{n\geq 1}$ of compact open subgroups K_n of G such that $\bigcap_n K_n = 1$. As $(\rho, C_c^\infty(G))$ is a smooth representation, we have

$$C_c^\infty(G) = \bigcup_{n\geq 1} {}^{K_n} C_c^\infty(G).$$

For each $n \geq 1$, from the above lemma, there exists a unique right G-invariant functional I_n on ${}^{K_n} C_c^\infty(G)$ which maps the characteristic function of K_n to $(K_1 : K_n)^{-1}$ (this number exists due to Proposition 1.2.2.2). We have I_{n+1} restricted to ${}^{K_n} C_c^\infty(G)$ is I_n. Indeed, since $K_{n+1} \subset K_n$ so ${}^{K_n} C_c^\infty(G) \subset^{K_{n+1}} C_c^\infty(G)$. With I_{n+1} restricted to ${}^{K_n} C_c^\infty(G)$, we obtain an element of $\text{Hom}_G({}^{K_n} C_c^\infty(G), \mathbb{C})$, so in order to show it is precisely I_n, we need to show this map sends h_{K_n} to $(K_1 : K_n)^{-1}$. Following proof of above lemma, we now that I_{n+1} is defined by sending characteristic function h_g on $K_{n+1}g$ to $(K_1 : K_{n+1})^{-1}$. Hence, the characteristic function h_{K_n} on $K_n = \bigcup_{g\in K_n/K_{n+1}} K_{n+1}g$ is $h_{K_n} = \sum_{g\in K_n/K_{n+1}} h_g$. As $h_g \mapsto (K_1 : K_{n+1})^{-1}$ under I_{n+1}, $h_{K_n} \mapsto (K_n : K_{n+1})/(K_1 : K_{n+1}) = (K_1 : K_n)^{-1}$, as desired.

Thus, the family $\{I_n\}$ gives a functional on $C_c^\infty(G)$ of the required kind. The unique statement is immediate, as any right Haar integral I' when restricted to ${}^{K_n} C_c^\infty(G)$ must satisfy uniqueness condition as in lemma. □

Remark 2.3.1.4. The lemma also implies that, if we view $C_c^\infty(G)$ as smooth representation of G under right translation, then $\dim \text{Hom}_G(C_c^\infty(G), C) = 1$. The proof follows by applying the lemma and by decomposing $C_c^\infty(G)$ as in previous proof.

The functional I of the proposition is a right Haar integral on G. One can produce a left Haar integral in exactly the same way. Alternatively, one can proceed as follows:

Corollary 2.3.1.5. For $f \in C_c^\infty(G)$, define $\check{f} \in C_c^\infty(G)$ by $\check{f}(g) = f(g^{-1}), g \in G$. The functional

$$I' : C_c^\infty(G) \to \mathbb{C}, I'(f) = I(\check{f}),$$

is a left Haar integral on G. Moreover, any left Haar integral on G is of the form cI', for some $c > 0$.

The uniqueness statement follows by observing that if J left Haar integral, then $f \mapsto J(\check{f})$ is a right Haar integral.

Let I be a *left* Haar integral on G. Let $S \neq \emptyset$ be compact open subset of G and Γ_S be its characteristic function. We define

$$\mu_G(S) = I(\Gamma_S).$$

Then $\mu_G(S) > 0$ (based on what we define I previous proposition) and the measure μ_G satisfies $\mu_G(gS) = \mu_G(S), g \in G$ (since $\lambda_g \Gamma_S = \Gamma_{gS}$ and I being left Haar measure implies $I(\lambda_g f) = I(f)$). One refers to μ_G as *left Haar measure* on G. The relation with the integral is expressed via traditional notation

$$I(f) = \int_G f(g) d\mu_G(g), f \in C_c^\infty(G).$$

So right invariance under this notation is

$$\int_G f(xg) d\mu_G(x) = \int_G f(x) d\mu_G(x), g \in G.$$

Further traditional abbreviations are frequently permitted, in particular,

$$\int_G \Gamma_S(x) f(x) d\mu_G(x) = \int_S f(x) d\mu_G(x).$$

Definition 2.3.1.6. The group G is *unimodular* if any left Haar integral on G is a right Haar integral.

2.3.2 Haar integral on some extensions of $C_c^\infty(G)$

section3.2

One can extend the domain of Haar integration, much as in the classical case of Lebesgue measure. We outline examples of what we will need.

First, one can integrate more general functions. For example, let f function on G invariant under left translation by a compact open subgroup K of G. Let μ_G be left Haar measure on G. If the series

$$\sum_{g \in K\backslash G} \int_{Kg} |f(x)| d\mu_G(x)$$

converges, so does the series without the absolute value, and we put

$$\int_g f(x) d\mu_G(x) = \sum_{g \in K\backslash G} \int_{Kg} f(x) d\mu_G(x).$$

The result does not depend on the choice of K, and this extended Haar integral has the translation invariance property of the original.

Question 2.3.2.1. How to construct such extension? I.e. how to define $\int_{Kg} |f(x)| d\mu_G(x)$ when $f \notin C_c^\infty(G)$?

Next, let G_1, G_2 be locally profinite groups, and set $G = G_1 \times G_2$. Then G is locally profinite. An element $\sum_{1 \le i \le r} f_i^1 \otimes f_i^2$ of the tensor product $C_c^\infty(G_1) \otimes C_c^\infty(G_2)$ gives a function on G

$$\Phi(g_1, g_2) = \sum_i f_i^1(g_1) f_i^2(g_2).$$

Then $\Phi \in C_c^\infty(G)$ and this process gives an isomorphism $C_c^\infty(G_1) \times C_c^\infty(G_2) \to C_c^\infty$. Let μ_j be left Haar mesure on $G_j, j = 1, 2$. There is then a unique left Haar measure μ_G on G such that

$$\int_G f_1 \otimes f_2(g) d\mu_G(g) = \int_{G_1} f_1(g_1) d\mu_1(g_1) \int_{G_2} f_2(g_2) d\mu_2(g_2),$$

for $f_i \in C_c^\infty(G_i)$. One writes $\mu_G = \mu_1 \otimes \mu_2$.

For a general $f \in C_c^\infty(G)$, the function

$$f(g_1) = \int_{G_2} f(g_1, g_2) f \mu_2(g_2)$$

lies in $C_c^\infty(G_1)$. We have

$$\int_g f(g) d\mu_G(g) = \int_{G_1} f_1(g_1) d\mu_1(g_1),$$

and symmetrically (indeed, if f is of the form $f_1 \otimes f_2$ then this is obvious, but note such functions span $C_c^\infty(G)$).

Next, let V complex vector space, and consider space $C_c^\infty(G; V)$ of locally constant, compactly supported functions $f : G \to V$. This space is isomorphic to $C_c^\infty(G) \otimes V$ in the obvious way: a tensor $\sum_j f_i \otimes v_i$ gives the function $g \mapsto \sum_i f_i(g) v_i$. If μ_G is left Haar measure on G, there is a unique linear map $I_V : C_c^\infty(G; V) \to V$ such that

$$I_V(f \otimes v) = \int_G f(g) d\mu(g) \cdot v.$$

We write

$$I_V(\phi) = \int_G \phi(g) d\mu_G(g), \phi \in C_c^\infty(G; V).$$

This has the same invariance properties as the Haar integral on scalar-valued functions.

2.3.3 Modular character

section3.3

Remark 2.3.3.1. This remark is served for quick referencing. Essentially what you learn after this section on modular character is just that:

$$\int_G \delta_G(x)^{-1} f(xg) d\mu_G(x) = \int_G \delta_G(x)^{-1} f(x) d\mu_G(x),$$

where μ_G is left Haar measure.

Let μ_G be a left Haar measure on G. For $g \in G$, consider the functional $C_c^\infty(G) \to \mathbb{C}$:

$$f \mapsto \int_G f(xg) d\mu_G(x).$$

This is a left Haar integral on G, so there is a unique $\delta_G(g) \in \mathbb{R}_+^\times$ such that

$$\delta_G(g)\int_G f(xg)d\mu_G(x) = \int_G f(x)d\mu_G(x),$$

for all $f \in C_c^\infty(G)$ [5]. The function δ_G is a homomorphism $G \to \mathbb{R}_{>0}^\times$. It is trivial if G is abelian; more generally, δ_G is trivial if and only if G is unimodular. Furthermore, δ_G is a character of G. One calls δ_G the *modular/module character* of G.

Check the above. We show δ_G is a group homomorphism. It suffices to show $\delta_G(g_2)\delta_G(g_1) = \delta_G(g_1)\delta_G(g_2) = \delta_G(g_1g_2)$. We have

$$\begin{aligned}\delta_G(g_2)\delta_G(g_1)\int_G f(xg_1g_2)d\mu_G(x) &= \delta_G(g_2)\delta_G(g_1)\int_G (\rho_{g_2}f)(xg_1)d\mu_G(x),\\ &= \delta_G(g_2)\int_G(\rho_{g_2}f)(x)d\mu_G(x),\\ &= \int_G f(x)d\mu_G(x).\end{aligned}$$

Next, we show δ_G is continuous by showing that $\delta_G^{-1}(1)$ is open. Take f to be a characteristic function of some open compact subgroup K of G then $f(xg) = f(x)$ for all $g \in K$ implying

$$\int_G f(xg)d\mu_G(x) = \int_G f(x)d\mu_G(x)$$

for all $g \in K$ and hence, $\delta_G(g) = 1$ for all $g \in K$, as desired. Thus, δ_G is a character of G.

Next, we show the functional

$$f \mapsto \int_G \delta_G(x)^{-1}f(x)d\mu_G(x), f \in C_c^\infty(G),$$

is a right Haar integral on G. Indeed, note that $f/\delta_G \in C_c^\infty(G)$ so by definition of δ_G, we have

$$\delta_G(g)\int_G \delta_G(xg)^{-1}f(xg)d\mu_G(x) = \int_G \delta_G(x)^{-1}f(x)d\mu_G(x).$$

As $\delta_G(xg) = \delta_G(x)\delta_G(g)$ so we obtain

$$\int_G \delta_G(x)^{-1}(\rho_g f)(x)d\mu_G(x) = \int_G \delta_G(x)^{-1}f(x)d\mu_G(x),$$

showing that the desired funtional is right Haar integral on G. Hence, if δ_G is trivial then every left Haar integral is a right Haar integral on G. Conversely, if G is unimodular, How to show that δ_G is trivial? □

[5] Just want to take note how other source define δ_G. Daniel Bump in his Lie Groups' book defined δ_G via $\int_G f(g^{-1}xg)g\mu_G(x) = \delta_G(g)\int_G f(x)d\mu_G(h)$. In this case $\delta_G(x)d\mu_G(x)$ is the right Haar measure, while in our case, it's $\delta_G(x)^{-1}d\mu_G(x)$, which will be proved later

The mnemonic $d\mu_G(xg) = \delta_G(g)d\mu_G(x)$ may be found helpful. How?

Remark 2.3.3.2. We have already observed that δ_G is trivial on any compact open subgroup of G. In particular, if G is compact, then $\delta_G = 1$ and G is unimodular. In the general case, any character $G \to \mathbb{R}^\times_{>0}$ is trivial on compact subgroups, since $\mathbb{R}^\times_{>0}$ has only the trivial compact subgroup.

2.3.4 Haar measure on $H\backslash G$

section3.4

Let H be closed subgroup of G, with module δ_H. Let $\theta : H \to \mathbb{C}^\times$ be character of H. We consider the space of functions $f : G \to \mathbb{C}$ which are G-smooth under right translation, compactly supported modulo H and satisfy

$$f(hg) = \theta(h)f(g), h \in H, g \in G.$$

We call this space $C_c^\infty(H\backslash G, \theta)$, and view it as a smooth G-space via right translation ρ. (Note $C_c^\infty(H\backslash G, \theta) = c\text{-Ind}_H^G\theta$, but this characterization is not helpful).

To spell this out (since I already forgot about these conditions), we have

1. From section 2.2.5, f being compactly supported modulo H means supp $f = \overline{\{g \in G : f(g) \neq 0\}}$ is compact in $H\backslash G$, i.e. supp $f \subset HC$ for some compact set C in G.

2. In section 2.3.1 $C_c^\infty(H\backslash G, \theta)$ is G-smooth under right translation ρ where the translation is defined as: $\rho f : x \mapsto f(xg); x, g \in G; f \in C_c^\infty(H\backslash G, \theta)$.

3. Last condition is $f(hg) = \theta(h)f(g), h \in H, g \in G$.

Proposition 2.3.4.1. Let $\theta : H \to \mathbb{C}^\times$ be character of H. The following are equivalent:

1. There exists a non-zero linear functional $I_\theta : C_c^\infty(H\backslash G, \theta) \to \mathbb{C}$ such that $I_\theta(\rho_g f) = I_\theta(f)$, for all $g \in G$.

2. $\theta\delta_H = \theta_G|_H$.

With these conditions hold, the functional I_θ is uniquely determined up to constant factor.

Proof. Let μ_G, μ_H be left Haar measure on G, H respectively. We define G-map $C_c^\infty(G) \to C_c^\infty(H\backslash G, \theta)$, denote $f \mapsto \widetilde{f}$, by

$$\widetilde{f}(g) = \int_H \theta\delta_H(h)^{-1} f(hg) d\mu_H(h).$$

This map satisfies $\widetilde{\lambda_k f} = \delta_H\theta(k)^{-1}\widetilde{f}$, for $k \in H$ and $f \in C_c^\infty(G)$. It is surjective. We check these first:

1. First, we show $\widetilde{f} \in C_c^\infty(H\backslash G, \theta)$. First, we check $\widetilde{f}$ is compactly supported modulo H. Indeed, note f being compactly support so supp $f \subset C$ for some open compact C of G, implying supp $\widetilde{f} \in HC$.

2. To show $C_c^\infty(H\backslash G, \theta)$ is G-smooth representation, we will skip this step.

3. To show $\widetilde{f}(h'g) = \theta(h')\widetilde{f}(g)$ for $h \in H, g \in G$, we have
$$\begin{aligned}\widetilde{f}(h'g) &= \int_H \theta\delta_H(h)^{-1} f(hh'g) d\mu_H(h),\\ &= \theta(h') \int_H \delta_H(h)^{-1}\theta(hh')^{-1} f(hh'g) d\mu_H(h),\\ &= \theta(h') f(g) \text{ (since } \delta_H \text{ modular character).}\end{aligned}$$

4. Finally, we check $\widetilde{\lambda_k f} = \delta_H\theta(k)^{-1}\widetilde{f}$. We have
$$\begin{aligned}\widetilde{\lambda_k f}(g) &= \int_H \theta\delta_H(h)^{-1}(\lambda_k f)(hg) d\mu_H(h),\\ &= \int_H \theta\delta_H(h)^{-1} f(k^{-1}hg) d\mu_H(h),\\ &= \theta\delta(k)^{-1} \int_H \theta\delta_H(k^{-1}h)^{-1} f(k^{-1}hg) d\mu_H(h),\\ &= \theta\delta_H(k)^{-1}\widetilde{f}(g)\end{aligned}$$

We prove the map is surjective. If K compact open subgroup of G, the space $C_c^\infty(G)^K$ is spanned by characteristic functions of cosets $gK, g \in G/K$ (since each $f \in C_c^\infty(G)^K$ is compactly supported, and supp f is disjoint union of gK for $g \in G/K$ so $f(g) \neq 0$, implying supp f is finite union of gK for $g \in G/K$ so $f(g) \neq 0$; this shows that f is span by characteristic function f_{gK} on cosets $gK, g \in G/K$). One the other hand, each coset HgK supports, at most, a one-dimensional space of functions in $C_c^\infty(H\backslash G, \theta)^K$ and these subspaces span $C_c^\infty(H\backslash G, \theta)^K$ [6]. The map $f \mapsto \widetilde{f}$ is surjective on K-fixed functions. It is therefore surjective.

Suppose that the space $C_c^\infty(H\backslash G, \theta)$ admits a functional I_θ of the required kind. The map $f \mapsto I_\theta(\widetilde{f})$ is then a nontrivial G-homomorphism $(C_c^\infty(G), \rho) \to \mathbb{C}$ [7].

However, the space $\mathrm{Hom}_G(C_c^\infty(G), \mathbb{C})$ has dimension 1 (remark from section 2.3.1), and it is spanned by a right Haar integral. Hence, up to constant factor of I_θ, the right Haar integral $\delta_G^{-1} d\mu_G : C_c^\infty(G) \to \mathbb{C}$ must factors through quotient map (since it's surjective as proven) $C_c^\infty(G) \to C_c^\infty(H\backslash G, \theta)$. Hence, the kernel of

[6] To explain this in details: With similar argument, note that for $f \in C_c^\infty(H\backslash G, \theta)^K$ then $f(HgK) \neq 0$ iff $f(g) \neq 0$, meaning supp f is finite union of HgK so each $f \in C_c^\infty(H\backslash G, \theta)^K$ is spanned by $\widetilde{f}_{HgK}$ with $g \in H\backslash G/K$ ($\widetilde{f}_{HgK}(g) = 1$ and 0 everyelse in $H\backslash G/K$). Finally, note that for fix $g \in H\backslash G/K$, a linear combination of characteristic functions $f_{hgK} \in C_c^\infty(G), h \in H$ must be mapped to $\mathbb{C}\widetilde{f}_{HgK} \subset C_c^\infty(H\backslash G, \theta)$

[7] Here's a quick check: if we call this map φ then we need $\varphi(\rho_g f) = \rho_g\varphi(f)$. Note it's $\mathbb{C}$ so $\rho_g\varphi(f) = \varphi(f)$. Hence, we need $I_\theta(\widetilde{\rho_g f}) = I_\theta(\widetilde{f})$ but $\widetilde{\rho_g f} = \rho_g\widetilde{f}$ since $f \mapsto \widetilde{f}$ is a G-map and from knowing I_θ, we are done.

$C_c^\infty(G) \to C_c^\infty(H\backslash G,\theta)$ must lie inside kernel of $\delta_G^{-1}d\mu_G$. By universal property for the quotient map $C_c^\infty(G) \to C_c^\infty(H\backslash G,\theta)$, we have

$$\begin{tikzcd} C_c^\infty(G) \arrow[d, two heads, "f\mapsto\widetilde{f}"'] \arrow[dr, "\delta_G^{-1}d\mu_G"] & \\ C_c^\infty(H\backslash G,\theta) \arrow[r, "\exists!\varphi"'] & \mathbb{C} \end{tikzcd}$$

where φ is defined as $\varphi(\widetilde{f}) = \int_G \delta_G(g)^{-1}f(g)d\mu_G(g)$. This follows

$$\begin{aligned} \delta_H\theta(h^{-1})\varphi(\widetilde{f}) &= \varphi(\widetilde{\lambda_h f}), \\ &= \int_G \delta_G(g)^{-1}(\lambda_h f)(g)d\mu_G(g), \\ &= \delta_G(h^{-1})\int_G \delta_G(h^{-1}g)^{-1}f(h^{-1}g)d\mu_G(g), \\ &= \delta_G(h^{-1})\varphi(\widetilde{f}) \end{aligned}$$

As this is true for all $f \in C_c^\infty(G)$ and all $h \in H$ so we obtain (2), as desired.

For the converse, we take function $f \in C_c^\infty(G)$ such that $\widetilde{f} = 0$. The function f is fixed by a compact open subgroup K (see section 2.3.1), and it is enough enough to treat the case where supp $f \subset HgK$ for some $g \in K$. Thus, f is finite linear combination of the characteristic function on cosets $h_igK, h_i \in H$. The condition $\widetilde{f} = 0$ then amounts to

$$\mu_H(H\cap gKg^{-1})\sum_i \theta\delta_H(h_i)^{-1}f(h_ig) = 0.$$

since $\theta\delta_H$ is trivial on the compact subgroup $H\cap gKg^{-1}$ of H. On the other hand,

$$\begin{aligned} \int_G f(x)\delta_G(x)^{-1}d\mu_G(x) &= \sum_i \int_K \theta\delta(h_igk)^{-1}f(h_igk)d\mu_G(k), \\ &= \mu_G(K)\delta_G(g)^{-1}\sum_i \delta_G(h_i)^{-1}f(h_ig). \end{aligned}$$

If (2) holds then we find the kernel of quotient map $C_c^\infty(G) \to C_c^\infty(H\backslash G,\theta)$ lies in the kernel of the right Haar integral $\delta_G^{-1}d\mu_G$ so by universal property of the quotient map, we obtain a nonzero G-map from $C_c^\infty(H\backslash G,\theta) \to \mathbb{C}$. $\square$

When the conditions of the proposition hold, the character θ takes only positive real values (since δ_H, δ_G only take positive real values). Let $f \in C_c^\infty(G)$ satisfy $f(g) \geq 0$, for all $g \in G$; we then have $\widetilde{f}(g) \geq 0$ for all g. Consequently:

Corollary 2.3.4.2. Suppose the condition of the proposition holds. There is then a non-zero linear functional I_θ on $C_c^\infty(H\backslash G,\theta)$ such that

1. $I_\theta(\rho_g f) = I_\theta(f)$ for $f \in C_c^\infty(H\backslash G,\theta), g \in G$;
2. $I_\theta(f) \geq 0$ for $f \in C_c^\infty(H\backslash G,\theta)$, $f \geq 0$.

These conditions determine I_θ uniquely, up to a positive constant factor.

One habitually uses a notation like

$$I_\theta(f) = \int_{H\backslash G} f(g) d\mu_{H\backslash G}(g), f \in C_c^\infty(H\backslash G, \theta),$$

and calls $\mu_{H\backslash G}$ a semi-invariant measure on $H\backslash G$. (Since $\theta = \delta_H^{-1}\delta_G|_H$ is uniquely determined, there is no real need to refer to it again).

2.3.5 Duality theorem of induced rep with compact induction

Let G be locally profinite group and H closed subgroup of G. Put

$$\delta_{H\backslash G} = \delta_H^{-1}\delta_G|_H : H \to \mathbb{R}_+^\times.$$

Theorem 2.3.5.1 (Duality theorem)**.** Let $\dot{\mu}$ be a positive semi-invariant measure on $H\backslash G$. Let (σ, W) be smooth representation of H. There is a natural isomorphism

$$\left(\text{c-Ind}_H^G \sigma\right)^\vee \cong \text{Ind}_H^G \delta_{H\backslash G} \otimes \check{\sigma}.$$

depending only on the choice of $\dot{\mu}$.

Proof. We view the $\delta_{H\backslash G} \otimes \check{\sigma}$ on the same space $\check{W}$ as $\check{\sigma}$. Let $(\check{w}, w) \mapsto \langle \check{w}, w\rangle$ be the evaluation pairing $\check{W} \times W \to \mathbb{C}$. Let $\phi \in \text{c-Ind}\sigma$, $\Phi \in \text{Ind}\delta_{H\backslash G} \otimes \check{\sigma}$, and consider the function

$$f : g \mapsto \langle \Phi(g), \phi(g)\rangle, g \in G.$$

□

DO THIS PART

2.3.6 The Hecke Algebra

If G is a finite group, the concept of a representation of G is essentially identical to that of module over group algebra $\mathbb{C}[G]$. This relation extends to smooth representations of locally profinite group, but only with a suitable definition of 'group algebra'.

To avoid technical complications, we impose the following:

Hypothesis. Unless otherwise stated, we assume that G is unimodular.

We fix a Haar measure μ on G. For $f_1, f_2 \in C_c^\infty(G)$, we define

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g) d\mu(x).$$

The function $(x, g) \mapsto f_1(x) f_2(x^{-1}g)$ lies in $C_c^\infty(G \times G)$ (section 2.3.2) so $f_1 * f_2 \in C_c^\infty(G)$. Similarly, for $f_i \in C_c^\infty(G)$, the integral expressing $f_1 * (f_2 * f_3)(g)$ is that of function from $C_c^\infty(G \times G)$, so we can manipulate formally

$$\begin{aligned} f_1 * (f_2 * f_3)(g) &= \int\int f_1(x)(f_2 * f_3)(x^{-1}g)d\mu(x) \\ &= \int\int f_1(x)f_2(y)f_3(y^{-1}x^{-1}g)d\mu(y)d\mu(x) \\ &= \int\int f_1(x)f_2(x^{-1}y)f_3(y^{-1}g)d\mu(y)d\mu(x) \\ &= \int\int f_1(x)f_2(x^{-1}y)f_3(y^{-1}g)d\mu(x)d\mu(y) \\ &= (f_1 * f_2) * f_3(g). \end{aligned}$$

The binary operation $*$, called *convolution* is thus associative. The pair

$$\mathcal{H}(G) = (C_c^\infty(G), *)$$

is an associative $\mathbb{C}$-algebra called *Hecke algebra* of G. In general, $\mathcal{H}(G)$ has no unit element; it is commutative if G is commutative.

Remark 2.3.6.1. The algebra structure on $\mathcal{H}(G)$ with $*$ depends on the choice of Haar measure μ. However, suppose we have two Haar measures μ, ν, giving rise to two algebra structures $\mathcal{H}_\mu(G), \mathcal{H}_\nu(G)$ on $C_c^\infty(G)$. There is a constant $c > 0$ such that $\nu = c\mu$. The map $f \mapsto c^{-1}f$ is then an algebra isomorphism $\mathcal{H}_\mu(G) \to \mathcal{H}_\nu(G)$.

While, in general, $\mathcal{H}(G)$, has no unit element, it does have family of *idempotents* elements. For example, let K be compact open subgroup of G, define function $e_K \in \mathcal{H}(G)$ by

$$e_K(x) = \begin{cases} \mu(K)^{-1} & \text{if } x \in K, \\ 0 & \text{if } x \notin K. \end{cases}$$

Proposition 2.3.6.2. 1. The function e_K satisfies $e_K * e_K = e_K$.

2. A function $f \in \mathcal{H}(G)$ satisfies $e_K * f = f$ iff $f(kg) = f(g)$ for all $k \in K, g \in G$.

3. The space $e_K * \mathcal{H}(G) * e_K$ is a subalgebra of $\mathcal{H}(G)$, with unit element e_K.

Proof. First, consider integral

$$e_K * e_K(g) = \int_G e_K(x)e_K(x^{-1}g)d\mu(x)$$

If $g \notin K$ then either $x \notin K$ or $x^{-1}g \notin K$. Hence, the integral is 0. If $g \in K$, then the integral is 0 when $x \notin K$ and is $\mu(K)^{-2}$ when $x \in K$.

For (2), for $f \in \mathcal{H}(G), k \in K, g \in G$, we have

$$\begin{aligned} e_K * f(kg) &= \int_G e_K(x)f(x^{-1}kg)d\mu(x), \\ &= \int_G e_K(kx)f(x^{-1}g)d\mu(x), \\ &= \int_G e_K(x)f(x^{-1}g) = (e_K * f)(g). \end{aligned}$$

If $e_K * f = f$ then we find $f(kg) = f(g)$ from this. Conversely, if f is left K invariant, then $(e_K * f)(g) = f(g)$, which implies $f(kg) = f(g)$ from the above.

For (3), it is not hard to see that $e_K * \mathcal{H}(G) * e_K$ is a subalgebra of $\mathcal{H}(K)$. $e_K * \mathcal{H}(G) * e_K$ consists of biinvariant functions (from above identity) so (2) implies e_K is the identity. $\square$

We note that $e_K * \mathcal{H}(G) * e_K$ is the space of $f \in \mathcal{H}(G)$ satisfying $f(k_1 g k_2) = f(g)$ for $g \in G, k_1, k_2 \in K$. The often write $\mathcal{H}(G,K) = e_K * \mathcal{H}(G) * e_K$.

Let M be a left $\mathcal{H}(G)$-module: it will be convenient to denote the module action by $(f, m) \mapsto f * m$, for $f \in \mathcal{H}(G), m \in M$. We say that M is *smooth* if $\mathcal{H}(G) * M = M$. Since $\mathcal{H}(G)$ is the union of its subalgebras $e_K * \mathcal{H}(G) * e_K$, the module

Chapter 3

Representation of $\mathrm{GL}_2(\mathbb{F}_q)$

We work out irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ of intertible 2×2 matrices over finite field $\mathbf{k}$

3.1 Linear groups $\mathrm{GL}_2(F)$

F denotes arbitrary field. We recall some basic facts about group $G = \mathrm{GL}_2(F)$.

3.1.1 Subgroups N, B, T, Z

section5.1

G has some important subgroups:

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in G \right\}, N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in G \right\}, T = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\}, Z = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right\}. \tag{3.1.1.1}$$

{subgroups_gl2}

B called *standard Borel subgroup* of G, N is unipotent radical of B, T is the standard split maximal torus in G. Z is the center of G, canonically isomorphic to $F^\times$.

Question 3.1.1.1. Explain above terminologies.

We have $B = T \ltimes N$ with N normal subgroup of B, implying $B/N \cong T$. This is due to $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix}$.

3.1.2 Bruhat decomposition

The *Bruhat decomposition* $G = G \cup BwB$ where w denotes the permutation matrix $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. That is, $\{1, w\}$ is set of representatives for coset space $B \backslash G / B$. This is true since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & a/c \\ 0 & 1 \end{pmatrix} w \begin{pmatrix} -c & -d \\ 0 & b - ad/c \end{pmatrix}, c \neq 0.$$

BwB are precisely matrices in G with nonzero $(2,1)$-entry. Since $B = NT = TN$ and w normalizes T (i.e $wT = Tw$), we have $BwB = N(Tw)TN = N(wT)TN = NwTN = NwB$ and similarly $BwB = BwN$. Moreover, the map

$$B \times N \to BwN, (b, n) \mapsto bwn,$$

is bijective (obviously surjective, to show injective, solve for $bwn = w$ to find $b = n = 1$).

3.2 Representations of $\mathbf{GL}_2(\mathbb{F}_q)$

Denote $\mathbb{F}_q$ finite field with q elements. We classify the irreducible (complex) representations of finite group $G = \mathrm{GL}_2(\mathbb{F}_q)$. (I will follow Etingof's book Representation theory for this part).

3.2.1 Conjugacy classes of $GL_2(\mathbb{F}_q)$

Consider a matrix $A \in G = \mathrm{GL}_2(\mathbb{F}_q)$. If A has two distinct eigenvalues x, y then by Jordan canonical form, A is conjugate (or similar) to $\begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}$. If A has repeated eigenvalue x then by Jordan canonical form, A is conjugate to either $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ or $\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$.

If A doesn't have eigenvalues in $\mathbb{F}_q$, i.e. it has characteristic polynomial irreducible over $\mathbb{F}_q$. Here we suppose q has characteristic other than 2, then every quadratic extension of $\mathbb{F}_q$ can be written as $\mathbb{F}_q(\sqrt{\varepsilon})$ where $\varepsilon \in \mathbb{F}_q \setminus \mathbb{F}_q^2$. Over this field, A will have eigenvalues $\alpha = x + \sqrt{\varepsilon}y$ and $\overline{\alpha} = x - \sqrt{\varepsilon}y$ with $x, y \in \mathbb{F}_q, y \neq 0$ and corresponding eigenvectors $v, \overline{v}$ where $Av = \alpha v, A\overline{v} = \overline{\alpha}$. Note that conjugacy (simiarity) of matrices indicates they represent the same linear map, which means if we change basis $e_1 = v + \overline{v}, e_2 = \varepsilon(v - \overline{v})$ then A is conjugate to $\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} \in G$. Due to the steps above, if $A, B \in G$ are matrices with irreducible characteristic polynomial f_A, f_B over $\mathbb{F}_q$, then A is conjugate to B iff $f_A = f_B$, i.e. they have same set of eigenvalues over $\mathbb{F}_q(\sqrt{\varepsilon})$, which determines $x, \pm y$ uniquely (where (x, y) and $(x, -y)$ are considered the same since set of eigenvalues of the two are the same). Therefore, the number of conjugacy classes of such type of matrices is equal to number of pairs $(x, \pm y)$ with $y \neq 0$. There are q ways for x and $\frac{q-1}{2}$ ways for y, giving $\frac{1}{2}q(q-1)$ such conjugacy classes.

Representatives	Size of class	Number of class
Scalar $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$	1	$q-1$ (one for every nonzero x)
Parabolic $\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$	q^2-1 (elements commuting with this one are $\begin{pmatrix} t & u \\ 0 & t \end{pmatrix}, t \neq 0$)	$q-1$ (one for every nonzero x)
Hyperbolic $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, x \neq y$	q^2+q (elements commuting with this one are $\begin{pmatrix} t & 0 \\ 0 & u \end{pmatrix}, t, u \neq 0$)	$\frac{1}{2}(q-1)(q-2)$ ($x, y \neq 0$ and x, y)
Elliptic $\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}, x \in \mathbb{F}_q, y \in \mathbb{F}_q^\times, \varepsilon \in \mathbb{F}_q \backslash \mathbb{F}_q^2$ (i.e. those whose characteristic polynomial over $\mathbb{F}_q$ is irreducible)	q^2-q (reason described below)	$\frac{1}{2}q(q-1)$ (matrices with y and $-y$ are conjugate)

table_conjugacyClass_GL2

Table 3.1: Conjugacy classes of $GL_2(\mathbb{F}_q)$

To calculate size of conjugacy class, note that for conjugacy class C with $x \in C$ then $|C_G(x)| \cdot |C| = |G|$ where $C_G(x)$ centralizer of x, i.e. elements commuting with x. We know $|G| = (q^2-1)(q^2-q)$ so it suffices to count number of elements commuting with x.

For the case where A has irreducible characteristic polynomial over $\mathbb{F}_q$, to count number of elements in conjugacy class of A, we use the basis $\{v, \overline{v}\}$ (i.e. A is then $\begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix}$), then matrices commuting with A will have the form $\begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix}$ for all $\lambda \in \mathbb{F}_{q^2}^\times$, implying that number of matrices commuting with A is q^2-1. Hence, size of conjugacy class of A is $\frac{|G|}{q^2-1} = q^2-q$.

See https://www3.nd.edu/~sevens/gl2f.pdf for another discussion about conjugacy classes or https://www.imsc.res.in/~amri/html_notes/notesap1.html#x8-28000A for general $GL_n(\mathbb{F}_q)$.

3.2.2 Characters of N in representation of B

section6.2

First, characters $\chi : N \to \mathbb{C}^\times$ produces a 1-dimensional representation of N as $\chi(n)v = \chi(n)v$ for all $n \in N$. Consider representation π of B that contains a 1-dimensional representation of N obtained via character χ. Note that taking T-

conjugation gives $gNg^{-1} = N$, more precisely

$$gn_cg^{-1} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix} = \begin{pmatrix} 1 & acb^{-1} \\ 0 & 1 \end{pmatrix}.$$

This follows $\pi(gn_cg^{-1})v = \chi(acb^{-1})v$ or $\pi(n_c)w = \chi(acb^{-1})w$ where $w = \pi(g^{-1})v$, meaning we have found (may be another) 1-dimensional representation of N within π. The question is how many distinct 1-dimensional representation of N can appear in π?

Group N of upper triangular unipotent matrices in G is isomorphic to additive group of $\mathbf{k}$, via the map $x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. Hence we study characters of N by studying characters of $\mathbf{k}$. If we fix a non-trivial character ψ of $\mathbf{k}$, the function $a\psi : x \mapsto \psi(ax), x \in \mathbf{k}$, ranges over all characters of $\mathbf{k}$ as a ranges over $\mathbf{k}$. This point of view implies that if π contains a non-trivial character of N then it contains all non-trivial characters of N.

3.2.3 1-dimensional representations

3.2.4 Principal series representation

We first consider a method of constructing irreducible representation of G.

Let χ_1, χ_2 be characters of $\mathbb{F}_q^\times$. We form the character

$$\chi = \chi_1 \otimes \chi_2 : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto \chi_1(a)\chi_2(b)$$

of T, which we then regard this as character of B, trivial on N via the quotient $B \to B/N \cong T$. In other words, $\chi \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \chi_1(a)\chi_2(c)$. Define $V_{\lambda_1,\lambda_2} = \mathrm{Ind}_B^G \mathbb{C}_\chi$ where $\mathbb{C}_\chi$ is 1-dimensional representation of B in which B acts by λ. We have $\dim V_{\lambda_1,\lambda_2} = |G|/|B| = q+1$.

Irreducible components in $\mathrm{Ind}_B^G \mathbb{C}_\chi$ are classified as follow:

Lemma 3.2.4.1. Let π irreducible representation of G. The following are equivalent:

1. π is equivalent to a G-subspace of $\mathrm{Ind}_B^H \chi$, for some character χ of T;
2. π contains the trivial character of N, i.e. when view π as representation of N then there exists trivial subrepresentation of N.

Proof. The representation π contains the trivial character of N if and only if there exists $v \in V$ so $\pi(N)v = v$. This v generates representation σ of B which is irreducible. Its character χ is trivial on N. By Frobenius Reciprocity, we have $\mathrm{Hom}_G(\pi, \mathrm{Ind}_B^G \sigma) \cong \mathrm{Hom}_B(\mathrm{Res}_B^G \pi, \sigma)$. Note that σ is contained in π when viewing π as representation of B. Therefore, $\mathrm{Hom}_B(\mathrm{Res}_B^G \pi, \sigma)$ is nontrivial, implying there's a copy of π inside $\mathrm{Ind}_B^G \sigma$. □

Thus, we classify irreducible representaions of V that contains trivial representation of N by analyzing $\text{Ind}_B^G \mathbb{C}_\chi$:

Theorem 3.2.4.2. 1. $\lambda_1 \neq \lambda_2 \implies V_{\lambda_1,\lambda_2}$ is irreducible.

2. $\lambda_1 = \lambda_2 = \mu \implies V_{\lambda_1,\lambda_2} = \mathbb{C}_\mu \oplus W_\mu$ where W_μ is a q-dimensional irreducible representation of G.

3. $W_\mu \cong W_\nu$ iff $\mu = \nu$; $V_{\lambda_1,\lambda_2} = V_{\lambda_1',\lambda_2'}$ iff $\{\lambda_1, \lambda_2\} = \{\lambda_1', \lambda_2'\}$ (in the second case $\lambda_1 \neq \lambda_2, \lambda_1' \neq \lambda_2'$).

Proof. From formula to characters of induced representation, we have

$$\text{Tr}_{V_{\lambda_1,\lambda_2}}(g) = \frac{1}{|B|} \sum_{a \in G, aga^{-1} \in B} \lambda(aga^{-1}).$$

- If $g = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$, the RHS is $\lambda(g)\frac{|G|}{|B|} = \lambda_1(x)\lambda_2(x)(q+1)$ as $aga^{-1} = g$.

- If $g = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$, the RHS equals to $\lambda(g) \cdot 1$ since $aga^{-1} \in B \implies a \in B$.

- If $g = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$, the RHS is $(\lambda_1(x)\lambda_2(y) + \lambda_1(y)\lambda_2(x)) \cdot 1$ since $aga^{-1} \in B$ implies $a \in B$ or a is an element of B multiply by the transposition matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

- If $g = \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}, 0 \neq y$, the RHS evaluates to 0 because matrices of this type do not have eigenvalues over $\mathbb{F}_q$ and thus cannot be conjugated into B (B is upper-triangular matrix so has eigenvalue in $\mathbb{F}_q$).

By using the above and the fact that $\chi_i(x)$ is a root of unity, i.e. $|\chi_i(x)| = 1$ (as

$\chi_i : \mathbb{F}_q^\times \to \mathbb{C}^\times$ and elements of $\mathbb{F}_q^\times$ have finite degree), we obtain

$$\begin{aligned}|G|\langle \chi_{V_{\lambda_1,\lambda_2}}, \chi_{V_{\lambda_1,\lambda_2}}\rangle &= \sum_{g\in G} \chi_{V_{\lambda_1,\lambda_2}}(g)\overline{\chi_{V_{\lambda_1,\lambda_2}}(g)},\\ &= \sum_{x\in\mathbb{F}_q^\times} 1\cdot |\lambda_1(x)|^2|\lambda_2(x)|^2(q+1)^2 + \sum_{x\in\mathbb{F}_q^\times}(q^2-1)|\chi_1(x)|^2|\chi_2(x)|^2\\ &+ \sum_{x\neq y,\{x,y\}\in\binom{\mathbb{F}_q^\times}{2}} (q^2+q)\cdot|\lambda_1(x)\lambda_2(y)+\lambda_1(y)\lambda_2(x)|^2\cdot 1,\\ &= (q-1)(q+1)^2+(q^2-1)(q-1)\\ &+(q^2+q)\sum_{x\neq y,\{x,y\}\in\binom{\mathbb{F}_q^\times}{2}} \big(|\lambda_1(x)|^2|\lambda_2(y)|^2+|\lambda_1(y)|^2|\lambda_2(x)|^2\\ &+\lambda_1(x)\lambda_2(y)\overline{\lambda_1(y)\lambda_2(x)}+\overline{\lambda_1(x)\lambda_2(y)}\lambda_1(y)\lambda_2(x)\big),\\ &= (q-1)(q+1)^2+(q^2-1)(q-1),\\ &+2(q^2+q)\binom{q-1}{2}+(q^2+q)\sum_{(x,y)\in(\mathbb{F}_q^\times)^2,x\neq y}\lambda_1(x)\lambda_2(y)\overline{\lambda_1(y)\lambda_2(x)}.\end{aligned}$$

If $\lambda_1=\lambda_2=\mu$ then the last term is equal to $(q^2+q)(q-2)(q-1)$, and the total in this case is $2|G|$, so $\langle \chi_{V_{\lambda_1,\lambda_2}}, \chi_{V_{\lambda_1,\lambda_2}}\rangle = 2$. We have $\mathbb{C}_\mu \subset \mathrm{Ind}_B^G\mathbb{C}_\mu$ since

$$\mathrm{Hom}_G(\mathbb{C}_\mu, \mathrm{Ind}_B^G\mathbb{C}_\mu) = \mathrm{Hom}_B(\mathbb{C}_\mu,\mathbb{C}_\mu)=\mathbb{C}.$$

due to Frobenius reciprocity. Therefore, combining with $\langle \chi_{V_{\mu,\mu}}, \chi_{V_{\mu,\mu}}\rangle = 2$, we find $\mathrm{Ind}_B^G\mathbb{C}_\mu = \mathbb{C}_\mu\oplus W_\mu$ where W_μ is irreducible; and the character of W_μ is different for distinct values of μ, proving that W_μ are distinct. [1]

If $\lambda_1\neq\lambda_2$ then let $z=xy^{-1}$. Note that $\chi(x^{-1})=\overline{\chi(x)}$ so the last term in the summation equals to

$$(q^2+q)\sum_{x\in\mathbb{F}_q^\times, z\neq 1}\frac{\lambda_1}{\lambda_2}(z) = (q-1)(q^2+q)\sum_{z\neq 1}\frac{\lambda_1}{\lambda_2}(z).$$

Since $\sum_{z\in\mathbb{F}_q^\times}\frac{\lambda_1}{\lambda_2}(z)=0$ because sum of all roots of unity (since $\lambda_1\neq\lambda_2$) of given order $m>1$ is zero, which makes the last term to be

$$-(q^2+q)(q-1)\frac{\lambda_1}{\lambda_2}(1) = -(q^2+q)(q-1).$$

This gives $\langle \chi_{V_{\lambda_1,\lambda_2}}, \chi_{V_{\lambda_1,\lambda_2}}\rangle = 1$ so V_{λ_1,λ_2} is irreducible. To show $V_{\lambda_1,\lambda_2}\cong V_{\lambda_2,\lambda_1}$, observe that we only change the character of hyperbolic element up to permutation of λ_i.

Any quick way to show $W_\mu \ncong W_\nu$ and $V_{\lambda_1,\lambda_2}\ncong V_{\lambda_1',\lambda_2'}$? Look for Maschke's theory. □

The representations $W_\mu, V_{\lambda_1,\lambda_2}, \lambda_1\neq\lambda_2$ are called *principal series representations*.

[1] Here is a bit abuse of notation, at the beginning we mention $\lambda_1=\lambda_2=\mu$ so $\mu:\mathbb{F}_q^\times\to\mathbb{C}^\times$ but later on $\mu : G\to\mathbb{C}^\times$ viewed as $\mu\otimes\mu$.

3.2.5 Cuspidal representations

An irreducible representation of G not containing the trivial character of N is called *cuspidal*. Such a representation must contain some non-trivial character of N.

Let $\mathbf{l}/\mathbf{k}$ quadratic extension. The only non-trivial $\mathbf{k}$-automorphism of field $\mathbf{l}$ is the Frobenius automorphism $x \mapsto x^q$. A character θ of $\mathbf{l}^\times$ is called *regular* if $\theta^q \neq \theta$.

By choosing a $\mathbf{k}$-basis of $\mathbf{l}$, we identify G with $\mathbf{k}$-linear invertible endomorphism on $\mathbf{l}$. The natural action of $\mathbf{l}^\times$ on $\mathbf{l}$ (i.e. for $a \in \mathbf{l}^\times$ then $f : \mathbf{l} \to \mathbf{l}$ defined as $v \mapsto av$ is an invertible linear map) thus gives an embedding of $\mathbf{l}^\times$ in G. We hence identify $\mathbf{l}^\times$ with subgroup E of G. Note that any element of G with irreducible characteristic polynomial is conjugate to an element of E. Indeed, for q odd, $\mathbf{l} = \mathbb{F}_q(\sqrt{\varepsilon})$ with $\varepsilon \in \mathbb{F}_q \setminus \mathbb{F}_q^2$ and we choose $1, \sqrt{\varepsilon}$ to be the basis, which immediately identify $x + y\sqrt{\varepsilon} \in \mathbf{l}^\times$ with $\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}$. So essentially, one can view E as $\left\{ \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} ; (x,y) \neq (0,0) \right\}$ (at least when computing character later on).

Let θ be regular character of E and ψ a non-trivial character of N. We define a character θ_ψ of ZN by

$$\theta_\psi : \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} u \mapsto \theta(a)\psi(u), a \in \mathbf{k}^\times, u \in N.$$

We observe that, by section 3.2.2, the representation $\mathrm{Ind}_{ZN}^G \theta_\psi$ is, up to equivalence, independent of the choice of ψ, hence the name θ_ψ. Explain this

Theorem 3.2.5.1. Let θ regular character of E and ψ a nontrivial character of N.

1. The virtual representation
$$\pi_\theta = \mathrm{Ind}_{ZN}^G \theta_\psi - \mathrm{Ind}_E^G \theta$$
is an irreducible representation of G, of dimension $q - 1$.

2. Let θ_1, θ_2 be regular characters of E; then $\pi_{\theta_1} \cong \pi_{\theta_2}$ if and only if $\theta_2 = \theta_1$ or $\theta_2 = \theta_1^q$.

3. Every irreducible cuspidal representation of G is of the form π_θ, for some regular character θ of E.

Proof. Part (3) follows from first two parts by counting dimensions. We prove the first two parts by computing characters. We find:

$$\begin{aligned}
\mathrm{tr}\pi_\theta \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} &= (q-1)\theta(x), x \in \mathbf{k}^\times; \\
\mathrm{tr}\pi_\theta \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix} &= -\theta(x), x \in \mathbf{k}^\times; \\
\mathrm{tr}\pi_\theta \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} &= 0, x \neq y, xy \neq 0; \\
\mathrm{tr}\pi_\theta \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} &= \theta(x + y\sqrt{\varepsilon}) + \theta(x + y\sqrt{\varepsilon})^q.
\end{aligned}$$

We will check the mentioned character table: First, we have by character formula for induced representation:

$$\mathrm{tr}\pi_\theta(g) = \frac{1}{|ZN|} \sum_{a \in G, aga^{-1} \in ZN} \theta_\psi(aga^{-1}) - \frac{1}{|E|} \sum_{a \in G, aga^{-1} \in E} \theta(aga^{-1}).$$

1. If $g = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ then $aga^{-1} = g$ for all $a \in G$. This implies

   $$\mathrm{tr}\pi_\theta(z) = \frac{|G|}{|ZN|}\theta_\psi(z) - \frac{|G|}{|E|}\theta(z) = \left(\frac{|G|}{|ZN|} - \frac{|G|}{|E|} \right) \theta(z).$$

   Count $|E| = |\mathbf{l}^\times| = q^2 - 1$; $ZN = q(q-1)$ since $|N| = q, |Z| = q-1$; $|G| = (q^2-1)(q^2-q)$ so we obtain $(q-1)\theta(z)$, as desired.

2. If $g = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$ then

   $$\begin{aligned}
   aga^{-1} \in ZN &\iff \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \\
   &\iff \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix} = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \\
   &\iff x = y, a = xzd, c = 0.
   \end{aligned}$$

   As $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} xzd & b \\ 0 & d \end{pmatrix}$ invertible so $a = xzd \neq 0$. For given $g = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$, we find $y = x$ and $z \neq 0$, there are $q(q-1)$ choices for b, d, giving

   $$\sum_{a \in G, aga^{-1} \in ZN} \theta_\psi(aga^{-1}) = q(q-1)\theta(x) \sum_{z \neq 0} \psi(z) = -q(q-1)\theta(x),$$

   as $\{\psi(z) : z \in \mathbf{k}\}$ is set of all roots of unity of some degree $m > 1$ so their sum is 0, which means when we exclude $z = 0$, we get the above result.

   On the other hand, $aga^{-1} \notin E$ since element of E is conjugate to an element of irreducible characteristic over $\mathbf{k}$, so it is not conjugate of parabolic element g.

3. If $g = \begin{pmatrix} x & 0 \\ 0 & t \end{pmatrix}$ with $x \neq t$. With same argument, $aga^{-1} \notin E$. We have

$$\begin{aligned} aga^{-1} \in ZN &\iff \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \\ &\iff \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \\ &\implies t = x. \end{aligned}$$

Thus, $\mathrm{tr}\pi_\theta(g) = 0$.

4. If $g = \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}$ with $y \neq 0$. Since $ZN \subset B$ so no conjugate of g will be in ZN. On the other hand, as $aga^{-1} \in E$ for all $a \in G$. Why $\frac{1}{|E|} \sum_{a \in G, aga^{-1} \in E} \theta(aga^{-1}) = \theta(x + y\sqrt{\varepsilon}) + \theta(x + y\sqrt{\varepsilon})^q$

This character table gives part (2) straight away.
To prove (1), we have

$$\frac{1}{|G|} \sum_{g \in G} |\mathrm{tr}\pi_\theta(g)|^2 = \sum_{z \in Z} (q-1)$$

□

Chapter 4

Representations of $GL_2(F)$

Some reference:

1. https://www.math.ucla.edu/~jonr/eprints/padic.pdf
2. http://www.math.ubc.ca/~cass/research/pdf/godement-ias.pdf
3. http://www.ims.nus.edu.sg/Programs/liegroups/files/sing.pdf
4. http://www-users.math.umn.edu/~garrett/m/v/toy_GL2.pdf

4.1 GL$_n$ over local fields

4.1.1 Haar measure on $F, F^\times$

section7.4

We describe Haar measures attached to various locally profinite groups. We start with field F.

Lemma 4.1.1.1. The vector space $C_c^\infty(F)$ is spanned by the characteristic functions of sets $a + \mathfrak{p}^m, a \in F, m \in \mathbb{Z}$.

Proof. The characteristic functions $a + \mathfrak{p}^m$ lies in $C_c^\infty(F)$. Let $\Phi \in C_c^\infty$. Since Φ has compact support, which means it can be finite cover of $\mathfrak{p}^k$, implying there exists $n \in \mathbb{Z}$ such that supp$\Phi \subset \mathfrak{p}^n$. Also Φ is locally constant, meaning Φ is fixed under translation by a compact open subgroup of F, hence by $\mathfrak{p}^m$, for some $m \in \mathbb{Z}$ (the argument follows similar in section 2.3.1). Thus, Φ is a linear combination of characteristic functions of $a + \mathfrak{p}^m, a \in \mathfrak{p}^n/\mathfrak{p}^m$. $\square$

If Φ_0 denotes the characteristic function of $\mathfrak{o}$ and μ is Haar measure on F, we have $\mu(\mathfrak{o}) = \int_F \Phi_0(x) d\mu(x) = x_0$ for some $c_0 > 0$. If Φ_1 is the characteristic function of coset $a + \mathfrak{p}^b, a \in F, b \in \mathbb{Z}$ then

$$\int_F \Phi_1(x) d\mu(x) = c_0 q^{-b}.$$

Proof of this claim. Note that we have a sequence of open compact subgroups $\mathfrak{o} \supset \mathfrak{p} \supset \mathfrak{p}^2 \supset \cdots$ of F such that $\bigcap_n \mathfrak{p}^n = 0$. Following proof of existence of Haar integral in section 2.3.1, we know that if $\mu(\mathfrak{o}) = c_0$ then there exists right Haar integral sending characteristic function of $\mathfrak{p}^n$ to $c_0(\mathfrak{o} : \mathfrak{p}^n)^{-1} = c_0 q^{-n}$ where $q = (\mathfrak{o}/\mathfrak{p}) = |\mathfrak{o}/\mathfrak{p}|$. Note the Haar integral is invariant under translation so it also send characteristic functions of $a + \mathfrak{p}^n$ to $c_0 q^{-n}$. □

Now take $\Phi \in C_c^\infty(F)$ and $y \in F^\times$. Using the identity above, we find

$$\int_F \Phi(xy) d\mu(x) = \|y\|^{-1} \int_F \Phi(x) d\mu(x),$$

where, we recall, $\|y\| = q^{-v_F(y)}$.

Check this. Since Φ is linear combination of characteristic functions of $a + \mathfrak{p}^n$, it suffices to show the above identity for Φ characteristic function of $\mathfrak{p}^n$. As $y \in F^\times$ so $y = u\varpi^m$ for $u \in \mathfrak{o}^\times$ and $m = v_F(y)$. Therefore, $\Phi(xy)$ is a characteristic function of $\mathfrak{p}^{n-m}$, resulting the integral as $c_0 q^{m-n} = \|y\|^{-1} \int_F \Phi(x) d\mu(x)$. This is also right-invariant since F is abelian. □

We accordingly define a measure $\mu^\times$ on $F^\times$ by $d\mu^\times(x) = d\mu(x)/\|x\|$, meaning the following. If $\Phi \in C_c^\infty(F^\times)$, the function $x \mapsto \|x\|^{-1}\Phi(x)$ (vanishing at 0) lies in $C_c^\infty(F)$ check this later, so we can put

$$\int_{F^\times} \Phi(x) d\mu^\times(x) = \int_F \Phi(x)\|x\|^{-1} d\mu(x), \Phi \in C_c^\infty(F^\times).$$

Check this define Haar integral on $F^\times$.

4.1.2 Haar measure on $G = \mathbf{GL}_2(F)$

section7.5

Matrix ring $A = M_2(F)$ (as additive group) is product of 4 copties of F and so Haar measure is obtained by taking tensor product of 4 copies of Haar measure on F (as mentioned in section 2.3.2).

Proposition 4.1.2.1. Let μ be a Haar measure on A. For $\Phi \in C_c^\infty(G)$, the function $x \mapsto \Phi(x)\|\mathrm{det} x\|^{-2}$ (vanishing $A \setminus G$) lies in $C_c^\infty(A)$. The functional

$$\Phi \mapsto \int_A \Phi(x)\|\det(x)\|^{-2} d\mu(x), \Phi \in C_c^\infty(G),$$

is a left and right Haar integral on G. In particular, G is unimodular.

Prove this later

4.1.3 Haar measure on B, N, T

section7.6

Since $N \cong F$ and $T \cong F^\times \times F^\times$ there is nothing more to say about them. We have $B = T \ltimes N$, we define a linear functional on space $C_c^\infty(B) = C_c^\infty(T) \otimes C_c^\infty(N)$ by

$$\Phi \mapsto \int_N \int_T \Phi(tn) d\mu_T(t) d\mu_N(n), \Phi \in C_c^\infty(B),$$

where μ_T, μ_N are Haar measure on T, N, respectively Here Bushnell and Henniart wrote $\int_T \int_N d\mu_T d\mu_N$ instead of $\int_N \int_T d\mu_T d\mu_N$ as above. One verifies that this functional is left B-invariant, so is left integral on B. We will denote this functional as

$$\Phi \mapsto \int_B \Phi(b) d\mu_B(b).$$

Check the integral is left-invariant. For $c = sm \in B$ where $s \in T, m \in N$. We need to show $\int_N \int_T \Phi(smtn) d\mu_T(t) d\mu_N(n) = \int_N \int_T \Phi(tn) d\mu_T(t) d\mu_N(n)$. Indeed, note $smtn = (sms^{-1})(st)n$ where $sms^{-1} \in N, st \in T$. View $\Phi(sms^{-1}stn)$ as functional $f(st)$ (evaluated at st) so when integrate with respect to t, due to invariance of μ_T, we have

$$\int_T \Phi(smtn) d\mu_T(t) = \int_T f(st) d\mu_T(t) = \int_T f(t) d\mu_T(t) = \int_T \Phi(sms^{-1}tn) d\mu_T(t).$$

Next, we view $\int_T \Phi(n'tn) d\mu_T(t)$ as functional $f(n'n)$ evaluated at $n'n$ so due to invariance of μ_N, we have

$$\begin{aligned}
\int_N \Phi(sms^{-1}tn) d\mu_T(t) d\mu_N(n) &= \int_N f(sms^{-1}n) d\mu_N(n), \\
&= \int_N f(n) d\mu_N(n), \\
&= \int_N \int_T \Phi(tn) d\mu_T(t) d\mu_N(n).
\end{aligned}$$

Thus, μ_B is indeed left-invariant. □

The Haar measure μ_B may be thought as tensor product, $\mu_B = \mu_T \otimes \mu_N$, but the two factors do not commute. This reflects the fact that group B is not unimodular, i.e. it has nontrivial modular character δ_B (section 2.3.3). Recall that $\|y\| = q^{v_F(y)}$.

Proposition 4.1.3.1. The modular character δ_B of group B is given by

$$\delta_B : tn \mapsto \|t_2/t_1\|, n \in N, t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \in T. \tag{4.1.3.1}$$

{eq:modularChar_borel}

Proof. Setting $c = sm, m \in N, s = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \in T$, we get

$$\int_B \Phi(bc) d\mu_B(b) = \int_N \int_T \Phi(tss^{-1}nsm) d\mu_T(t) d\mu_N(n).$$

We use the obvious isomorphism $N \to F$ to identify μ_N with certain Haar measure μ_F on F. For $\phi \in C_c^\infty(N)$, we then have

$$\begin{aligned}\int_N \phi(s^{-1}ns)d\mu_N(n) &= \int_F \phi\begin{pmatrix}1 & s_1^{-1}xs_2\\ 0 & 1\end{pmatrix} d\mu_F(x),\\ &= \|s_1s_2^{-1}\| \int_N \phi(n)d\mu_N(n).\end{aligned}$$

where the last part follows from discussion in section 4.1.1. Following the same argument as when we show μ_B is left-invariant, we have

$$\begin{aligned}\int_N\int_T \Phi(tss^{-1}nsm)d\mu_T(t)d\mu_N(n) &= \int_N\int_T \Phi(ts^{-1}ns)d\mu_T(t)d\mu_N(n),\\ &= \|s_1s_2^{-1}\| \int_N\int_T \Phi(tn)d\mu_T(t)d\mu_N(n).\end{aligned}$$

From the definition of modular character, the result follows, i.e.

$$\int_B \Phi(bc)d\mu_B(b) = \delta_B(c)^{-1}\int_B \Phi(b)d\mu_B(b).$$

□

4.2 Representations of Mirabolic Group

4.3 Jacquet Modules and Induced Representations

4.3.1 Jacquet functor

section8.1

Let (π, V) be smooth representation of N and let ϑ be a character of N. We denote $V(\vartheta)$ the linear subspace of V spanned by the vector $\pi(n)v - \vartheta(n)v, n \in N, v \in V$. We set $V_\vartheta = V/V(\vartheta)$.

If ϑ_0 is the trivial character of N, we denote $V(\vartheta_0) = V(N)$ and we write $V_{\vartheta_0} = V_N$. In this case, we find that $V(N)$, and hence V_N, are T-modules, as $\pi(t)(\pi(n)v - v) = \pi(n')v' - v'$ where $n' = t^{-1}nt \in N, v' = \pi(t)v$. This action of T defines a smooth representation (π_N, V_N), which is called *Jacquet module of V* [1].

On the other hand, if ϑ_0 nontrivial then V_ϑ is only Z-module of $\mathrm{GL}_2(F)$.

Lemma 4.3.1.1. Let μ_N be a Haar measure on N and ϑ a character of N.

1. Let (π, V) be smooth representation of N and $v \in V$. The vector v lies in $V(\vartheta)$ if and only if there is a compact open subgroup N_0 of N such that

$$\int_{N_0} \vartheta(n)^{-1}\pi(n)v\,d\mu_N(n) = 0. \tag{4.3.1.1}$$ {eq:jacquet}

2. The process $(\pi, V) \mapsto V_\vartheta$ is an exact functor from $\mathrm{Rep}(N)$ to $\mathrm{Rep}(Z)$ (or $\mathrm{Rep}(T)$ if ϑ is the trivial character). This is called *Jacquet functor*.

[1] some places define (π, V) to be representation of G or of B

Proof. We assume first that ϑ is the trivial character of N. The group $N \cong F$ is the union open compact subgroups, so if

$$v = \sum_{i=1}^{r} v_i - \pi(n_i)v_i \in V(N)$$

then there is a compact open subgroup N_0 of N containing all n_i's (union of finite compact is also compact). It suffices to show $\int_{N_0} \pi(n)(\pi(n')v - v)d\mu_N(n) = 0$ for $n' \in N_0, v \in V$, which is true since μ_N is Haar measure on N.

Conversely, if $v \in V$ and suppose $\int_{N_0} \pi(n)vd\mu_N(n)$ holds. There is open normal subgroup N_1 of compact N_0 such that $v \in V^{N_1}$ [2]. The space V^{N_1} carries a representtion of the finite group N_0/N_1 (since $\pi(n_1)v = v$ for all $v \in V^{N_1}, n_1 \in N_1$ so $\pi(gN_1)v = \pi(g)v$ is well-defined). Therefore, in the obvious notation, $V^{N_1} = V^{N_1}(N_0/N_1) \oplus V^{N_0}$ (see section 2.2.3, here V^{N_0} as elements in V^{N_1} fixed by action of N_0/N_1) and the map

$$\pi(e_{N_0}) : w \mapsto \mu_N(N_0)^{-1} \int_{N_0} \pi(n)wd\mu_N(n), w \in V^{N_1},$$

(here $\mu_N(N_0) = \int_{N_0} 1d\mu_N(n)$) is the N_0-projection $V^{N_1} \to V^{N_0}$ (is it is N_0-homomorphism as μ_N is invariant, it is surjective as any $v \in V^{N_0}$, as $\pi(n)v = v$ for $n \in N_0$ so $v \mapsto \mu_N(N_0)^{-1} \int_{N_0} \pi(n)vd\mu_N(n) = v$ so it is indeed a projection). Its kernel is then $V^{N_1}(N_0/N_1) \subset V(N)$ (i.e. as defined, $V^{N_1}(N_0/N_1)$ is span of $v - \pi(n_0N_1)v = v - \pi(n_0)v$ for $v \in V^{N_1}$). We have proved (1) for the trivial character ϑ of N.

Now let ϑ be arbitrary character of N, and consider representation (π', V') of N, where $V' = V$ and $\pi'(n) = \vartheta(n)^{-1}\pi(n)$. We then have $V(\vartheta) = V'(N)$ and so (1) follows in general.

We prove (2) for the case ϑ is trivial character. If ϑ is nontrivial, the proof is similar (Later): First, we check that if $f : V \to W$ is N-module homomorphism then $f(v - \pi_V(n)v) = f(v) - \pi_W(n)f(v)$, implying $f(V(N)) \subset W(N)$ so the induced map $V_N \to W_N$ is well-defined. Next, let

$$0 \longrightarrow V' \xrightarrow{i} V \xrightarrow{p} V'' \longrightarrow 0$$

be short exact sequence of N-modules. We show that the induced sequence

$$0 \longrightarrow V'_N \xrightarrow{i_N} V_N \xrightarrow{p_N} V''_N \longrightarrow 0$$

is exact. First, we show i_N is injective. Without loss of generality, suppose V' is submodule of V and i is the inclusion map. Suppose $v' \in \ker i_N, v' \neq 0$ then $v' = i(v') \in V(N)$. According to part (1), there exists open compact subgroup N_0 of N such that $\pi(e_{N_0})(v') = 0$, and by applying part (1) again, we find $v' \in V'(N)$. Thus, i_N is indeed injective. On the other hand, p_N is surjective as p is surjective.

[2] since (π, V) smooth so exists open compact subgroup K' of N_0 so $v \in V^{K'}$ and since N_0 is compact so N_0/K' finite so $N_1 = \bigcap_{g \in N_0/K'} gK'g^{-1}$ is open normal subgroup of N_1 where $v \in V^{N_1}$

To show exactness at V_N, i.e. $\mathrm{im} i_N = \ker p_N$. If $v \in V$ has image 0 in V''_N, i.e. $p(v) \in V''_N$. From part (1), there exists compact open subgroup N_0 of N such that $\pi''(e_{N_0})p(v) = 0$. Note that $\pi''(e_{N_0}) : V'' \to (V'')^{N_0}$ is a N_0-projection, and since p is a N-morphism, we find $0 = \pi''(e_{N_0})p(v) = p(\pi(e_{N_0})v)$ [3]. Due to exactness at V, as $\pi(e_{N_0}v) \in \ker p = \mathrm{im}\, i$, there exists $v' \in V'$ so $i(v') = \pi(e_{N_0})v$. To show $v \in \mathrm{im}\, i$, it suffices to show $\pi(e_{N_0})v = v$ in V_N, which is true since we have $V = V^{N_0} \oplus V(N_0)$ according to section 2.2.3 and the fact that $\pi(e_{N_0})$ is N_0-projection map. □

Remark 4.3.1.2. A better way to show exactness is first prove exactness of

$$0 \longrightarrow V'(N) \longrightarrow V'(N) \longrightarrow V''(N) \longrightarrow 0$$

then use Snake lemma, as in Bump's book.

Proposition 4.3.1.3. Let (π, V) be smooth representation of N, and let $v \in V, v \neq 0$. There exists a character ϑ of N such that $v \notin V(\vartheta)$.

Proof. We write

$$N_j = \begin{pmatrix} 1 & \mathfrak{p}^j \\ 0 & 1 \end{pmatrix}, j \in \mathbb{Z}. \tag{4.3.1.2}$$

Take $v \in V, v \neq 0$. We choose N_{j_0} such that N_{j_0} fixes v. For $j \leq j_0$, let V_j denote the N_j-space generated by v. Since N_j is compact, V_j is the direct sum of isotypic components V_j^η where η ranges over characters of N_j trivial on N_{j_0} (since N_j is abelian so any irreducible smooth representation of N_j is one-dimensional, as in section 2.2.6). As $v \in V_j$, v can be writte as linear combination of $\sum_\eta \alpha_\eta v_\eta$ over all η such that $\alpha_\eta \neq 0$. Since $v \neq 0$ so there exists character η_j such that $V^{\eta_j} \neq 0$ and $\alpha_{\eta_j} \neq 0$. With this and note that $\pi(n)v_\eta = \eta(n)v_\eta$ for all $n \in N_j$, we find that the integral $\int_{N_j} \eta_j(n)^{-1}\pi(n)vdn$ contains $\alpha_{\eta_j}\int_{N_j}\eta_j(n)^{-1}\pi(n)v_{\eta_j}vdn = \alpha_{\eta_j}\mu_N(N_j)v_{\eta_j} \neq 0$ so

$$\int_{N_j} \eta_j(n)^{-1}\pi(n)vdn \neq 0.$$

The N_{j-1}-space generated by $V_j^{\eta_j} \subset V_j = \mathrm{span}\,\{\pi(n_j)v : n_j \in N_j\}$ is contained in V_{j-1} (as $N_j \subset N_{j-1}$), so we may choose η_{j-1} such that $\eta_{j-1}|_{N_j} = \eta_j$ (i.e. apply above argument for N_{j-1}-space generated by $V_j^{\eta_j}$ since action of N_j on this space is via η_j). Since F can be written as union of $\mathfrak{p}^n$ where n can range over all integers at most some M, we find that there exists a character ϑ of N such that for all $j \leq j_0$, we have

$$\int_{N_j} \vartheta(n)^{-1}\pi(n)vdn \neq 0.$$

Therefore, from previous lemma, we find $v \notin V(\vartheta)$. □

[3] Since p is N-morphism, it will send $V^{N_0} \to (V'')^{N_0}$. Hence, if $v = u \oplus w$ where $u \in V^{N_0}$ (this is possible since N_0 compact open, meaning V is N_0-semisimple) then $p(v) = p(u) \oplus p(w)$ with $p(u) \in (V'')^{N_0}$. We have $\pi(e_{N_0})(v) = u$.

Corollary 4.3.1.4. Let (π, V) be smooth representation of N. If $V_\vartheta = 0$ for all characters ϑ of N, then $V = 0$.

Let (σ, W) be smooth representation of T. We view σ as smooth representation of B which is trivial on N, and form the smooth induced representation $\mathrm{Ind}_B^G \sigma$. If (π, V) is a smooth representation of G, Frobenius Reciprocity from section 2.2.4 gives an isomorphism

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_B^G \sigma) \cong \mathrm{Hom}_B(\pi|_B, \sigma).$$

However, σ is trivial on N so any B-homomorphism $\pi \to \sigma$ factors through quotient map $\pi \to \pi_N$ (recall π_N is the Jacquet module). We deduce

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}\sigma) \cong \mathrm{Hom}_T(\pi_N, \sigma). \tag{4.3.1.3}$$ {eq:Frobenius_jacquet}

This has the following consequence:

Proposition 4.3.1.5. Let (π, V) be an irreducible smooth representation of G. The following are equivalent:

1. The Jacquet module V_N is nonzero.
2. The representation π is isomorphic to a G-subspace of a representation $\mathrm{Ind}_B^G \chi$ for some character χ of T.

Proof. Suppose (2) holds. From 4.3.1.3 we get

$$\mathrm{Hom}_T(\pi_N, \chi) \cong \mathrm{Hom}_G(\pi, \mathrm{Ind}\chi) \neq 0,$$

so $\pi_N \neq 0$.

To prove $(1) \implies (2)$, choose $v \in V, v \neq 0$. Since V irreducible over G, any element of V is finite linear combination of $\pi(g)v$ of v, for various $g \in G$. Write $K = \mathrm{GL}_2(\mathfrak{o})$. The vector is fixed by a compact subgroup K' of K of finite index (since K is compact); let $\{v_1, \ldots, v_r\}$ be the distinct elements of the form $\pi(k)v, k \in K/K'$. In particular, $r \leq (K : K')$. Since $G = BK$, the elements $v_1, \ldots, v_r$ generate V over B, and their images under $\pi \to \pi_N$ generate V_N over T.

Thus, V_N is finitely generated as representation of T. We choose minimal generating set $\{u_1, \ldots, u_t\}, t \geq 1$, then the T-subspace U containing $u_1, \ldots, u_{t-1}$ is maximal T-subspace of V_N and therefore, V_N/U is irreducible representation of T. Since T is abelian, section 2.2.6 implies that V_N/U is a character χ. Thus, $\mathrm{Hom}_T(\pi_N, \chi) \neq 0$ (as there is projection with kernal U to χ) implying $\mathrm{Hom}_G(\pi, \mathrm{Ind}_B^G \chi) \neq 0$ and since π is irreducible, π is isomorphic to G-subspace of representation $\mathrm{Ind}_B^G \chi$. $\square$

An irreducible smooth representation (π, V) of G is called *supercuspidal* (or absolutely cuspidal) if V_N is zero. On the other hand, if $V_N \neq 0$, one says π is the *principal series*.

4.3.2 One-dimensional representation

4.3.3 Decomposing principal series

In previous section, we know irreducible representtions (π, V) of G with nonzero Jacquet module has a copy in $\text{Ind}_B^G\chi$ for some character χ of T. Hence, our task now is to try to decompose $\text{Ind}_B^G\chi$ into irreducible representations.

Let μ_N be Haar measure on N and $t \in T$. The measure $S \mapsto \mu_N(t^{-1}St)$ is the Haat measure $\delta_B(t)\mu_N$, for δ_B modular character, as proved in proof of (4.1.3.1):

$$\int_N f(txt^{-1})d\mu_N(x) = \delta_B(t)\int_N f(x)d\mu_N(x), f \in C_c^\infty(N).$$

As before, let $w = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ be the permutation matrix. If σ is a smooth representation of T, we can form the representation $\sigma^w : t \mapsto \sigma(wtw^{-1})$, and view it as representation of B which is trivial on N.

As in section 2.2.4, α_σ denotes the canonical B-map $\text{Ind}_B^G\sigma \to \sigma$ given by $f \mapsto f(1)$. In induces canonical T-map $(\text{Ind}_B^G\sigma)_N \to \sigma$ (since σ acts trivially on N so the previous map must factor through $\text{Ind}\sigma \to (\text{Ind}\sigma)_N$), which we will continue to denote a_σ.

Lemma 4.3.3.1 (Restriction-Induction lemma)**.** Let (σ, U) be smooth representation of T and set $(\sum, X) = \text{Ind}_B^G\sigma$. There is an exact sequence of representations of T:

$$0 \longrightarrow \sigma^w \otimes \delta_B^{-1} \longrightarrow \textstyle\sum_N \xrightarrow{\alpha_\sigma} \sigma \longrightarrow 0.$$

Proof. By definition in section 2.2.4, X is the space of G-smooth functions $f : G \to U$ such that $f(bg) = \sigma(b)f(g), b \in B, g \in G$. The canonical map $\alpha_\sigma : X \to U$ amounts to restriction of functions to B (i.e. under α_σ, $f \mapsto f|_B$). Set $V = \ker\alpha_\sigma$ then V can be viewed as smooth representation of B.

First, we show $\alpha_\sigma : X \to U$ is surjective. For $u \in U$, consider function $f_u : G \to U$ defined by $f(K) = u$ for all $k \in K$ and $f_u(bk) = \sigma(b)u$. As $G = BK$, we have $f_u(gk) = f_u(g)$ for all $g \in G, k \in K$. According to definition of Ind_B^G as in section 2.2.4, note that K is compact, we have $f_u \in \text{Ind}_B^G\sigma = X$. We also have $\alpha_\sigma : f_u \mapsto u$ so α_σ is surjective.

Since α_σ is surjective and $V = \ker\alpha_\sigma$, we find an exact sequence of smooth representations of B:

$$0 \longrightarrow V \longrightarrow X \xrightarrow{\alpha_\sigma} U \longrightarrow 0.$$

Due to exactness of Jacquet functor, and that $U_N = U$ since N acts trivially on U, we obtain exact sequence of representations of T as follow:

$$0 \longrightarrow V_N \longrightarrow (\textstyle\sum_N, X_N) \xrightarrow{\alpha_\sigma} (\sigma, U) \longrightarrow 0.$$

Thus, it suffices to identify the T-representation V_N with $\sigma^w \otimes \delta_B^{-1}$. We recall that $G = B \cup BwN$. A function $f \in X$ thus lies in V if and only if supp$f \subset BwN$. More precisely:

Lemma 4.3.3.2. Let $f \in X$, then $f \in V$ if and only if there is a compact open subgroup N_0 of N (depending on f) such that supp $f \subset BwN_0$.

Proof. A function $f \in X$ lies in V if and only if $f(1) = 0$. As $f(bg) = \sigma(b)f(g)$ so we find f vanishes on B. The identity for $x \neq 0$:

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix} w \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix} \in Bw \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}$$

implies that supp $f \subset BwN_0$, for some compact open subgroup N_0 of N. Why this argument follows from the above identity? Also there is a typo in Bushnell, Henniart book, where $-x^{-1}$ should be x^{-1} instead. □

Let $f \in V$; in view of the lemma how, we can define function $f_N : T \to U$ by

$$f_N(x) = \int_N f(xwn)dn = \sigma(x)f_N(1), x \in T$$

By section 4.3.1, the kernel of the map $f \mapsto f_N$ is $V(N)$ [4], and so $f \mapsto f_N(1)$ gives a bijective map $V_N \to U$ (injectivity follows from previous sentence, surjectivity follows from the fact $f_N(1)$ generates $f : T \to U$ via $f(x) = \sigma(x)f_N(1)$). Taking

[4]Let me notation-chasing to explain this sentence (it may be false though). Kernel of this map is all $f \in V$ such that $f_N(1) = 0$.

Recall $V = \{f : G \to U, f \in \mathrm{Ind}_B^G \sigma, f(1) = 0\}$. We define Haar integral I on $C_c^\infty(N, V)$ by (one can see that this map is right-invariant, but what about other conditions?)

$$(F : N \to V) \mapsto \left(I(f) = \int_N F(n)dn \in V : x \in G \mapsto \int_N F(n)(x)dn \in U \right)$$

By section 4.3.1, $f \in V(N)$ iff $\int_N \sum(n)fdn \in V$ is the zero map (also note that $\sum(n)f \in V$ if $f \in V$ since $\sum(n)f(1) = f(n) = \sigma(n)f(1) = 0$). From above Haar integral, this map is the map (recall $\sum(f)f : x \in G \mapsto f(xg)$)

$$\varphi = \int_N \sum(n)fdn : (x \in G) \mapsto \int_N (\sum(n)f)(x)dn = \int_N f(xn)dn \in U.$$

If this is the zero map then $\int_N f(xn)dn = 0$ for all $x \in G$, implying with $x = w$ then $f_N(1) = 0$, as desired.

Conversely, if $f_N(1) = 0$ then $\int_N f(wn)dn = 0$. Note we have $f \in V$ so $f_N(1) = 0$ so $\varphi(w) = 0$. Note that $\varphi \in V$ so $\varphi(b) = \sigma(b)\varphi(1) = 0$ Also $\varphi(bwn') = \sigma(b)\varphi(wn')$ and $\varphi(wn') = \int_N f(wn'n)dn = \int_N f(wn)dn = \varphi(w) = 0$. As $G = B \cup BwN$ so $\varphi(G) = 0$, as desired.

$t \in T$ and $f \in V$, we have

$$\begin{aligned}(tf)_N(x) &= \int_N f(xwnt)dn, \\ &= \int_N f(x(wtw^{-1})wt^{-1}nt)dn = \int_N f(xt^w w(t^{-1}nt))dn, \\ &= \delta_B(t^{-1}) \int_N f(xt^w wn)dn, \\ &= \delta_B^{-1}(t)\sigma^w(t)\left(f_N(x)\right), \\ &= (\sigma^w \otimes \delta_B^{-1})(t)(f_N(x)).\end{aligned}$$

Hence, with $x = 1$, we find $f \mapsto f_N(1)$ is a B-homomorphism $(\sum, V) \to (\sigma^w \otimes \delta_B^{-1}, U)$ and since $(\sum, V)$ is trivial on N so this induces T-isomorphism $V_N \cong \sigma^w \otimes \delta_B^{-1}$. □

4.3.4 Principal series irrep is admissible

section9.4

The irreducible representations of G exhibit a helpful finiteness property:

Proposition 4.3.4.1. Let (π, V) be an irreducible smooth representation of G which is not (super)cuspidal. The representation π is admissible.

Proof. By definition $V_N \neq 0$. By section 4.3.1, π is equivalent to a subrepresentation of $\mathrm{Ind}_B^G\chi$ for some character of T. It is enough to prove $\mathrm{Ind}\chi$ is admissible.

We fix compact open subgroup K of G. We want to show X^K is finite dimensional where $\mathrm{Ind}_B^G\chi = (\sum, X)$. WLOG, we may assume $K \subset K_0 = \mathrm{GL}_2(\mathfrak{o})$ (as one can shrink K to $K_1 = K \cap K_0$ and note $X^K \subset X^{K_1}$). The space X^K of K-fixed points in $\mathrm{Ind}\chi$ consists of functions $f : G \to \mathbb{C}$ satisfying

$$f(bgk) = \chi(b)f(g), b \in B, g \in G, k \in K.$$

We have $G = BK_0$, so the set $B\backslash G/K$ is finite, and each of double coset BgK supports, at most, a one-dimensional space of functions satisfying above condition (**??** OK need to go back and read this). □

Chapter 5

Representations of $GL_n(F)$

Some reference http://www.math.tifr.res.in/~dprasad/ictp2.pdf

5.1 Structure of $\mathbf{GL}_n(F)$

Center of GL_n is denoted Z.

In matrix manners, the diagonal torus T is subgroup of all diagonal matrices, the standard Borel subgroup B consists of all upper triangular matrices. The unipotent radical U consists of all upper triangular unipotent matrices. Note T normalizes U and B is semidirect product TU. Let W, the *Weyl group* of G, is the group $N_G(T)/T$ where $N_G(T)$ normalizer of T. For $G = GL_n(F)$, W is subgroup of all monomial matrices (one nonzero entry in each row and each column) so W can be identified with S_n.

In different view, by picking n-dimensional vector space V over F, $GL_n(F)$ can be identified with $GL(V)$. Define a *flag* $W_\bullet$ in V to be strictly increasing sequence of subspaces $W_0 \subset W_1 \subset \cdots \subset W_m = V$. Subgroup of $GL(V)$ that stabilizes flag $W_\cdot$, i.e. with the property that $gW_i = W_i$ for all i is called *parabolic subgroup* of G associated to flag $W_\bullet$.

If $\{v_1, \ldots, v_n\}$ basis of V then stabilizer of flag $\{(v_1) \subset (v_1, v_2) \subset \cdots \subset (v_1, \ldots, v_n)\}$ is called *Borel subgroup*. In the case of $GL(V)$, stabilizer of any two such (maximal) flags are conjugate under $GL(V)$.

If $W_\bullet = \{W_0 \subset W_1 \subset \cdots \subset W_m\}$ then inside the associated parabolic subgroup P, there exists normal subgroup N consisting of elements who operating trivially on W_{i+1}/W_i for $0 \leq i \leq m-1$, which is called unipotent radical of P. There is a semidirect product decomposition $P = MN$ with $M = \prod_{i=0}^{m-1} GL(W_{i+1}/W_i)$. The decomposition $P = MN$ is called *Levi decomposition* of P with M called *Levi subgroup* of P.

Let $K = GL_n(\mathfrak{o})$ denote subgroups of elements in G in $\mathfrak{o}$ and whose determinant is unit in $\mathfrak{o}$. This is a maximal open compact subgroup of $GL_n(F)$, as shown in following exercise.

Exercise 10. Let V be n-dimensional vector space over F. Let L be lattice on V, i.e. $\mathfrak{o}$-submodule rank n. Show that stabilizer of L is open compact subgroup of $G = GL(V)$. If

C is any open compact subgroup then there is lattice L such that C lies in stabilizer of L. Hence, up to conjugacy, K is the unique maximal open compact subgroup of $GL_n(F)$.

For every integer $m \geq 1$, the map $\mathfrak{o} \to \mathfrak{o}/\mathfrak{p}^m$ induces map $K \to GL_n(\mathfrak{o}/\mathfrak{p}^m)$. The kernel K_m of this map is called *principal congruence subgroup of level m*. We also define K_0 to be K. For all $m \geq 1$, we have $K_m = 1_n + \mathfrak{p}^m M_n(\mathfrak{o})$. K_m are open compact subgroups of G and gives basis of neighborhood at the identity.

Bruhat decomposition gives $G = \bigsqcup_{w \in W} BwB = \bigsqcup_{w \in W} BwU$. Proof use row reduction (on the left) and column reduction (on the right) by elementary operations.

Cartan decomposition gives, for $A = \{\mathrm{diag}(\Pi^{m_1}, \dots, \Pi^{m_n}) : m_i \in \mathbb{Z}_{\geq 0}, m_1 \leq \cdots \leq m_n\}$ then $G = \bigsqcup_{a \in A} KaK$.

Iwasawa decomposition gives $G = KB$.

Iwahori factorization gives for $m \geq 1$ then $K_m = (K_m \cap U^-)(K_m \cap T)(K_m \cap U)$ where U^- subgroup of all lower triangular unipotent matrices.

5.1.1 In language of algebraic groups

(Very roughly, so that I can read other things)

An *algebraic group* is an algebraic variety G defined over some field F with morphims $m : G \times G \to G$ and $\mathrm{inv} : G \to G$ becomes multiplication and inverse map on $G(E)$ making $G(E)$ of E-rational points (i.e. $G(E) = G \cap E^n$) into a group when E is commutative F-algebra. Affine algebraic group is when G is affine variety, example is multiplicative group G_m, with $G(E) = E^\times$.

A *torus* is group T that is isomorphic to direct product of G_m^k for some k. If the isomorphism is defined over F we say T *split* (over F).

For affine algebraic group G over F. By *representation* we mean morphism $\rho : G \to GL_n$ for some n such that $\rho : G(E) \to GL_n(E)$ group hom for any commutative F-algebra E.

By Jordan decomposition, $g \in G$ then exists $g_s, g_u \in G$ such that $g = g_s g_u = g_u g_s$, g_s semisimple, g_u unipotent. Hence $g \in G$ is called *semisimple* if $g = g_s$ and *unipotent* if $g = g_u$.

G is unipotent if all its elements are unipotnent. Group G has maximal normal unipotent subgroup U, called *unipotent radical*. If U is trivial then G is *reductive*. If it is reductive and has no nontrivial normal tori then G is *semisimple*. For example group of $\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right\}$ is not reductive since $\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}$ is normal unipotent subgroup. Group SL_n is semisimple. Group GL_n is reductive but not semisimple.

Maximal torus T is subgroup as large as possible such that T product of multiplicative groups. If T splits over F then G is *F-split*. All maximal tori in $G(F)$ are conjugate if F algebraically closed.

If G is F-split reductive group and T is F-split maximal torus, N normalizer of T then N/T is Weyl group W.

See Bump's note and Fiona Murnaghan for more info.

Chapter 6

Questions

Question 6.0.0.1. What is character of $\mathbb{Q}_p$ and $\mathbb{Q}_p^\times$.

https://kconrad.math.uconn.edu/blurbs/gradnumthy/characterQ.pdf

Question 6.0.0.2. Give a natural G map V_{λ_1,λ_2} to V_{λ_2,λ_1}. Keyword: intertwiner integral.

Question 6.0.0.3. What is the point of compact induction?

Bibliography

pAdicRepFM [1] http://www.math.toronto.edu/murnaghan/courses/mat1197/notes.pdf