Symmetric Functions

Toan Q. Pham

Friday 5^{th} June, 2020

Contents

Abstract

This is an essay from a reading course supervised by professor Ole Warnaar, where I learnt many mathematics that are new to me.

In this essay we study an object call symmetric functions. Historically, we have a certain family of symmetric functions that have applications in various places: Schur functions in representation of symmetric groups, of $\mathrm{GL}(n, \mathbb{C})$; Hall-Littlewood in study of Hall algebra, in study of Green's polynomial from Springer theory, in characters of $\mathrm{GL}(n, \mathbb{F}_q)$; Jack symmetric functions. All of these functions have a generalisation called Macdonald's function. Macdonald [10] defined such function P_λ by constructing a self-adjoint operator with respect to that product where P_λ will then be its eigenfunctions. Such choice of self-adjoint operator looks quite unnatural to us so in our essay, we try to go the other way around in order to explain why such scalar product arises. This is the main subject in Chapter I.

Also in Chapter I, we learn about the Hopf algebra structure of ring of symmetric functions, and then introduce a second scalar product, P_λ is also orthogonal to. This scalar product appears to be more natural.

In chapter 2, we generalise Schur polynomial to Weyl characters and we describe explicitly that the case of A_{n-1} root system leads to the Schur polynomial. There is also a similar construction of P_λ under this generalisation but we don't add this in our chapter as we are not quite comfortable with such construction yet.

In chapter 3, we learn about connection of symmetric functions in representation theory.

This essay is not complete, in the sense that we have not filled out all the details we would like to. Furthermore, we have encountered many other connections that we have not had the chance to learn before finalising this essay. In particular, these are certain buzzwords that we would like to know about, in random order: (Double Affine) Hecke Algebra; $\mathrm{GL}(n, \mathbb{F}_q)$ characters; Weyl-Kac character formula; (affine) Lie algebra or Kac-Moody Lie algebra; Gelfald pairs; universal characters; Diagonal harmonics; Crystals; Hilbert scheme; Springer theory; Jack symmetric functions; Green polynomials; spherical functions; zonal polynomials, k-Schur, ...

Chapter 1

Symmetric functions

1.1 Definition

First, we recall the definition of ring of symmetric functions given in [10]: Denote Λ_n^k to be $\mathbb{Z}$-module of symmetric polynomials of degree k in n variables $x_1, \ldots, x_n$ with coefficients over field F of characteristic 0. Define $\Lambda^k = \varprojlim_{k\geq 0} \Lambda_n^k$ of F-modules Λ_n^k relative to the homomorphism $\rho_{m,n}^k : \Lambda_m^k \to \Lambda_n^k$ for $k \geq 0, m \geq n$ obtainted by sending $x_i \mapsto 0$ for $n < i \leq m$ (if $m = n$ then this is the identity map). This maps are always surjective, and bijective for $m \geq n \geq k$. This follows that the projection $\rho_n^k : \Lambda^k \to \Lambda_n^k$ which obtained by letting $x_i \mapsto 0$ for $i > n$, is an isomorphism for all $n \geq k$. The ring of symmetric functions Λ over field F is defined to be $\Lambda = \bigoplus_{k\geq 0} \Lambda^k$. We define $\rho_n = \bigoplus_{k\geq 0} \rho_n^k : \Lambda \to \Lambda_n$ where $\Lambda_n = \bigoplus_{k\geq 0} \Lambda_n^k$ (this viewed as ring of symmetric polynomials over n variables). Since ρ_n^k is an isomorphism for $n \geq k$, so ρ_n is an isomorphism in degrees $k \leq n$ (i.e. if one restricts ρ_n to $\bigoplus_{k\leq n} \Lambda^k$).

This definition has few important points:

1. Since $\Lambda = \bigoplus_{k\geq 0} \Lambda^k$ so every symmetric function $f \in \Lambda$ is of finite degree.

2. $\rho_n : \Lambda \to \Lambda_n$ is an isomorphism when one restricts to $\bigoplus_{k\leq n} \Lambda^k$. Combining with previous observation, if one wants to derive identities involving symmetric functions $f \in \Lambda$ of finite degree, it is sufficient to do that for $\rho_n(f) = f(x_1, \ldots, x_n)$ over large enough n, i.e. we restrict it to finite number of variables to work with.

3. The choice of variables $x_1, x_2, \ldots$ is not important, i.e. replacing it with a different $y_1, y_2, \ldots$ will not change the definition of Λ. However, the role that the variables $x_1, x_2, \ldots$ play in this definition is rather important, in the sense that it gives descriptions of the maps $\rho_n^k, \rho_n, \rho_{m,n}^k$. In particular, the order of putting the variables is important because our maps is defined based on a certain variable in this order. Thus, one would question, whether there is a different way to define Λ without invoking the choice of variables $x_1, x_2, \ldots$ (or any choice at all)? As far as we are aware, there is an axiomatic definition

of Λ as a positive self-dual Hopf algebra with only one primitive element according to [7].

1.2 Symmetric functions via plethystic notation

We will reintroduce symmetric functions $f \in \Lambda$ via a different notation, different from our usual $f(x_1, \ldots, x_n)$. The purpose for this is that we want a description of our function and do manipulation on it without having access to the letters x_i. This is indeed what we do with polynomial, even when one does not be able to factor it. One writes polynomial $P(x)$ as $P(x) = \prod_{a \in A}(x - a)$, A being alphabet of zeros of $P(x)$. Now, $\prod_{a \in A}(x - a^2), \prod_{a \in A}(x - a)^2$, for example, are perfectly defined polynomials whose coefficients can be written in terms of coefficients of P, though they have been defined in terms of roots of P [1].

In order to not having access to specific x_i when describing our $f \in \Lambda$, we put all inputs x_i's for f into single input X, called *alphabet*, and view our function f as an operator acting on X. For example, X can be the set $\{x_1, \ldots, x_n\}$ or $\{x_1, \ldots, ...\}$ (but it can also be something else). For later convenience, we will use the notation $X = x_1 + x_2 + \cdots$ when refering to alphabet $\{x_1, \ldots\}$. We also use *plethystic brackets* $[.]$ as in $f[X]$ instead of $f(X)$ to distinguish between alphabet X and set X. We call this *plethystic notation* or *λ-ring notation*. For example, we write

$$f(X) = f(x_1, x_2, \ldots) = f[X] = f[x_1 + x_2 + \cdots]$$

How can we describe f without having access to *letters* x_i (letters mean just a term in our summation of alphabet X)? We want to do this by relying on other symmetric functions (or operators in general) we know of. This is the same as how we can write Schur functions s_λ as linear combination of monomial symmetric functions m_μ or as polynomial of power sums p_r. Note that we don't have to indicate whether the input set is $\{x_1, \ldots, x_n\}$ or $\{x_1, \ldots,\}$ because our relations are valid for any such choice of input. This certainly matches with our goal.

In order to follow above approach, to start off, we need a base step, i.e. we need to choose a symmetric function and specify how it acts on general alphabet X. Our chosen one is the complete symmetric function $p_r \in \Lambda$. We define

$$\begin{aligned} p_r[X+Y] &:= p_r[X] + p_r[Y], \\ p_r[XY] &:= p_r[X]p_r[Y], \\ p_r[X-Y] &:= p_r[X] - p_r[Y], \\ p_r[X] &:= p_r(X) = x_1^r + x_2^r + \cdots \text{ for } X = x_1 + x_2 + \cdots \end{aligned}$$

The point here is that we want to put enough definitions of action of p_r on general alphabet X so that it mimics the behaviour of our well-known power sums p_r. Indeed, for two countable alphabets X, Y, we define addition and Cartesian product

[1] this paragraph is copypasted from [8].

of such two alphabets as

$$X+Y=\sum_{x\in X}x+\sum_{y\in Y}y,$$
$$XY=\left(\sum_{x\in X}x\right)\left(\sum_{y\in Y}y\right),$$

then with the power sum $p_r\in\Lambda$, we have $p_r[X+Y]=p_r[X]+p_r[Y]$ and $p_r[XY]=p_r[X]p_r[Y]$.

Similarly, if $Y=y_1+y_2+\cdots$ then $p_r(-Y)=p_r(-y_1,-y_2,\dots)=(-1)^r p_r(Y)$ and we want to implement this in our alphabet [2] so we denote $\varepsilon X:=\{-x_1,-x_2,\dots\}$ for $X=x_1+x_2+\cdots$ and hence define

$$p_r[\varepsilon X]:=(-1)^r p_r[X]$$

for arbitrary alphabet X. For example, we obtain $p_r[-\varepsilon X]=(-1)^{r-1}p_r[X]$.

After defining such operations in alphabets, note that we can manipulate alphabets as if they are ordinary elements of a commutative ring [3]. For example, we can show that

$$p_r[(X-Y)+Y]=p_r[X],p_r[X(Y-Z)]=p_r[XY-XZ].$$

One can allow special division in alphabets, by viewing $1/(1-q)$ as alphabet $1+q+q^2+\cdots$ and defined

$$p_r\left[\frac{X}{1-q}\right]:=\frac{p_r[X]}{1-q^r}=p_r[X(1+q+q^2+\cdots)]^{4},r\geq 1.$$

Here $1-q$ inside the plethystic bracket should be view as plethystic minus between two alphabets $1,q$, each has only one letter. So this says that two alphabets $1-q$ and $1+q+q^2+\cdots$ are inverse of each other.

Since Λ is $\mathbb{Q}$-generated by the power sums $p_r, r\geq 1$, any symmetric function can be defined for general alphabet. In particular, $f[X]$ should be equal to $f(X)$ for any $f\in\Lambda$ where $X=x_1+x_2+\cdots$ or $X_n=x_1+\cdots+x_n$ More generally, if function f has a alphabet-free [5] description based on other operators which are defined "globally" (i.e. for any alphabet) then f can also be defined for any alphabet (one can simply take this description to be definition). Since we start off with p_r's, this follows that to verify any such global description, e.g. $e_n[A]=(-1)^n h_n[A]$, it is enough to check it locally for specific choice of (finite) alphabet A, in this case is for $A=x_1+\cdots+x_m$ where $n\leq m$.

[2]note that the minus sign $p_r(-Y)$ is the ordinary minus sign while the minus sign in $p_r[-Y]$ is not

[3]it seems there is a more general theory for noncommutative alphabets, see [8]

[4]note that the equal sign on the right can be obtained from what we define so far.

[5]I can't find a better word here

We have $\lambda_z, \sigma_z, \Psi_z$ are generating functions of elementary symmetric functions e_r, complete symmetric functions h_r and power sums p_r, respectively. We define them in general alphabets as

$$\Psi_z[X] := \sum_{r\geq 1} \frac{p_r[X]z^r}{r}, \sigma_z[X] := \sum_{r\geq 0} h_r[X]z^r, \lambda_z := \sum_{r\geq 0} e_r[X]z^r.$$

Below are some identities in plethystic notations which holds for general alphabets. There are more in Macdonald's book [10] so this is listed for the sake of referencing later:

$$\sigma_z[X] = \exp(\Psi_z[X]), \Psi_z[X] = \log(\sigma_z[X]), \tag{1.2.0.1}$$

$$s_{\lambda/\mu} := \det\left(h_{\lambda_i - i - \mu_j + j}\right)_{1\leq i,j\leq n}, \ n \geq l(\lambda), h_r = 0 \text{ for } r < 0 \tag{1.2.0.2}$$

$$s_{\lambda/\mu} = \det\left(e_{\lambda'_i - i - \mu'_j + j}\right)_{1\leq i,j\leq n}, n \geq l(\lambda), \tag{1.2.0.3}$$

$$s_{\lambda/\nu}[A+B] = \sum_{\mu:\nu\subseteq\mu\subseteq\lambda} s_{\mu/\nu}[A]s_{\lambda/\mu}[B], \tag{1.2.0.4}$$

$$\sigma_z[-A] = \lambda_{-z}[A], \tag{1.2.0.5}$$

$$\lambda_z[A+B] = \lambda_z[A]\lambda_z[B], \tag{1.2.0.6}$$

$$\sigma_z[A+B] = \sigma_z[A]\sigma_z[B], \tag{1.2.0.7}$$

$$e_n[A] = (-1)^n h_n[-A], \tag{1.2.0.8}$$

$$h_n[A+B] = \sum_{k=0}^{n} h_k[A]h_{n-k}[B], \tag{1.2.0.9}$$

$$e_n[A+B] = \sum_{k=0}^{n} e_k[A]e_{n-k}[B], \tag{1.2.0.10}$$

$$s_{\lambda/\mu}[-X] = (-1)^{|\lambda/\mu|} S_{\lambda'/\mu'}[X]. \tag{1.2.0.11}$$

We also have Pieri's formula

$$s_\mu h_n = \sum_{\lambda} s_\lambda \tag{1.2.0.12}$$

summed over all partitions λ so $\lambda - \mu$ is horizontal n-strip.

For $X = x_1 + x_2 + \cdots$ or $X_n = x_1 + \cdots + x_n$, we have

$$\lambda_z[X] = \prod_{x\in X}(1+zx), \sigma_z[X] = \prod_{x\in X}\frac{1}{1-zx}, \Psi_z[X] = \sum_{i=1}^{\infty}\sum_{x\in X} z^i x^i/i, \tag{1.2.0.13}$$

$$s_\lambda(x_1,\ldots,x_n) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{1\leq i,j\leq n}}{\det\left(x_i^{n-j}\right)_{1\leq i,j\leq n}}, n \geq l(\lambda), \tag{1.2.0.14}$$

$$s_{\lambda/\mu}(x_1,\ldots,x_n) = \sum_{T\in SSYT(\lambda/\mu,\cdot)} x^{\text{content}(T)}. \tag{1.2.0.15}$$

What is the use of such notation? The point here is that instead of working with symmetric functions over $\mathbb{Z}$ or $\mathbb{Q}$, we work with the field $\mathbb{Q}(q,t)$ where q,t are indeterminates. What plethystic notation does is that it will treat q,t just as input/variables of the function instead of as part of the coefficients, which brings us back to working with $\mathbb{Z}$ or $\mathbb{Q}$ where we know a lot about. This is seen in simple manipulation/definition such as $p_r\left[\frac{X}{1-q}\right] = \frac{p_r[X]}{1-q}$ where we bring q inside the plethystic brackets.

There is also a question of what the alphabets can be. One option we know is that the alphabets can be seen as ring of polynomials or rational functions over $\mathbb{Q}$, where we start with $p_r[x] = x^r$ for variable x and extend it to $p_r[f]$ where f is any rational function [8, 4]. We wonder whether there is any other alphabets that fit the descriptions (for example, any combinatorial alphabets).

There is another interpretation of $f[A]$ following [9], rather than viewing as f acting on A, first we view A as a ring homomorphism φ_A from Λ to certain λ-ring R. From what we understand, λ-ring R means R has copies of power sums that behaves exactly the same as the power sums of Λ, and what f_A does is it sends p_r to the corresponding power sums in R and generates images of other elements from there. Hence, $f[A]$ can be seens as the evaluation of φ_A at $f \in \Lambda$, i.e. $\varphi_A(f) \in R$. In this setting, the variables $x_1, x_2, \ldots,$ is seen as rank 1 element in λ-ring R. We refer to [6] for more about λ-rings.

1.3 Scalar product on Λ

The Schur function $s_\lambda[X]$ can be shown to be a dual basis with respect to the scalar product defined by

$$\langle p_\lambda p_\mu \rangle = \delta_{\lambda\mu} z_\lambda$$

where $z_\lambda = \prod_{i\geq 1} i^{m_i} m_i!$ where $m_i(\lambda)$ is the multiplicity of i in λ. The Hall-Littlewood function P_λ, which is a t-deformation of the Schur function, is a dual basis of $\Lambda_{\mathbb{Q}(t)}$ with respect to the scalar product defined by

$$\langle p_\lambda, p_\mu \rangle = z_\lambda p_\lambda \left[\frac{1}{1-t}\right] \delta_{\lambda\mu}$$

where in usual notation, $p_\lambda\left[\frac{1}{1-t}\right] = \prod_{i\geq 0}(1-t^{\lambda_i})^{-1}$. An explicit formula for $P_\lambda[X;t]$ is given in [10, p. 208] as

$$P_\lambda(x_1,\ldots,x_n;t) = \frac{1}{v_\lambda(t)} \sum_{w\in S_n} w\left(x_1^{\lambda_1}\cdots x_n^{\lambda_n} \prod_{i<j} \frac{x_i - tx_j}{x_i - x_j}\right)$$

where $v_\lambda(t) = \prod_{i\geq 0}\prod_{j=1}^{m_i} \frac{1-t^j}{1-t}$. Here we can see that by taking $t=0$, we obtain Schur polynomials.

The Macdonald function $P_\lambda[X;q,t]$ is a q,t-deformation of the the previous two functions, that is dual basis of $\Lambda_{\mathbb{Q}(q,t)}$ with respect to the scalar product

$$\langle p_\lambda, p_\mu \rangle = z_\lambda p_\lambda \left[\frac{1-q}{1-t}\right] \delta_{\lambda\mu}.$$

Macdonald's constructed [10, Chapter V] Macdonald's polynomials by defining certain self-adjoint operator E with respect to the scalar product.

In this section, we aim to get a more general constructions of symmetric functions with respect to a more general scalar product, where we replace $p_\lambda\left[\frac{1-q}{1-t}\right]$ with a general multiplicative element v_λ with respect to the scalar product similar to the above family of symmetric functions. We follow the approach taken by [2] and [5] via introducing raising operators and vertex operators to find a recursive descriptions of these functions.

Let F field of characteristic 0, let $\langle , \rangle$ be non-degenerate symmetric F-bilinear form on Λ_F with values in F, such that the homogeneous component Λ_F^n of Λ_F are pairwise orthogonal. If $(u_i), (v_i)$ are dual bases of Λ_F^n, i.e. $\langle u_i, v_j\rangle = \delta_{i,j}$ then

$$T_n(x|y) = \sum_i u_i(x)v_i(y)$$

is independent of the choice of dual bases (so that in particular, $T_n(x|y) = T_n(y|x)$). Denote the *metric tensor*

$$T(x|y) = \sum_{n\geq 0} T_n(x|y)$$

Let $\Delta : \Lambda_F \to \Lambda_F \otimes \Lambda_F$ be the diagonal map (or comultiplication) defined by $\Delta f = f(x, y)$.

A family (a_λ) of elements of a commutative monoid, indexed by partition λ, will be called *multiplicative* if $a_\lambda a_\mu = a_{\lambda\cup\mu}$ for each pair of partitions λ, μ. Equivalently, $a_\lambda = a_{\lambda_1}a_{\lambda_2}\cdots$ for each partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ where $a_r = a_{(r)}$ for each integer $r \geq 1$ and $a_0 = 1$. For example, the bases $(e_\lambda), (h_\lambda), (p_\lambda)$ of Λ_F (note that since F is of characteristic 0 so these elements are indeed in Λ_F) are multiplicative.

Following [10][p. 306], the following conditions on the scalar product $\langle , \rangle$ are equivalent:

1. Δ is the adjoint of multiplication,

$$\langle \Delta f, g \otimes h\rangle = \langle f, gh\rangle \tag{1.3.0.1}$$

   for all $f, g, h \in \Lambda_F$.

2. There exists multiplicative family (v_k) in F^* such that

$$\langle p_\lambda, p_\mu\rangle = \delta_{\lambda,\mu} z_\lambda v_\lambda \tag{1.3.0.2}$$

   for all partitions λ, μ where $z_\lambda = \prod_{i\geq 1} i^{m_i} m_i!$ where $m_i(\lambda)$ is multiplicity of i in λ.

3. There exists a character (i.e. F-algebra homomorphism) $\chi : \Lambda_F \to F$ such that $\chi(p_n) \neq 0$ for all $n \geq 1$ and

$$\langle f, g\rangle = \chi(f * g)$$

   for all $f, g \in \Lambda_F$, where $f * g$ is the internal product.

4. The basis Λ_F dual to (m_λ) is multiplicative.

5. The metric tensor T satisfies

$$T(x|y,z) = T(x|y)T(x|z)$$

Example 1.3.0.1. ⌟

Suppose all of these conditions hold, we have

$$\begin{aligned} T(x|y) &= \sum_\lambda z_\lambda^{-1} v_\lambda^{-1} p_\lambda(x) p_\lambda(y), \\ &= \exp\left(\sum_{n\geq 1} \frac{v_n^{-1}}{n} p_n(x) p_n(y)\right), \end{aligned}$$

Let

$$\exp\left(\sum_{r\geq 1} \frac{v_r^{-1}}{r} p_r z^r\right) = \sum_{r\geq 0} q_r z^r, \tag{1.3.0.3}$$

then we have

$$\begin{aligned} T(x|y) &= \prod_i \left(\sum_{n\geq 0} q_n(x) y_j^n\right), &(1.3.0.4)\\ &= \sum_\lambda q_\lambda(x) m_\lambda(y) &(1.3.0.5) \end{aligned}$$

From (1.3.0.3), we find

$$q_n = \sum_{\lambda\vdash n} \frac{1}{z_\lambda v_\lambda} p_\lambda. \tag{1.3.0.6}$$

Proof. We have

$$\exp\left(\sum_{r\geq 1} \frac{v_r^{-1}}{r} p_r z^r\right) = \prod_{r\geq 1} \exp\left(\frac{p_r z^r}{r v_r}\right) = \prod_{n\geq 1} \sum_{m_r=0}^{\infty} \frac{(p_r z^r)^{m_r}}{r^{m_r} v_r^{m_i} m_r!} = \sum_\lambda z_\lambda^{-1} v_\lambda^{-1} p_\lambda z^{|\lambda|}.$$

□

Example 1.3.0.2. If one defines $\sigma_z[X] = \exp\left(\sum_{r\geq 1} \frac{p_r[X]}{r} z^r\right)$ then if $v_\lambda = \prod_{i=1}^k \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}} = p_\lambda\left[\frac{1-q}{1-t}\right]$ we have

$$\exp\left(\sum_{r\geq 1} \frac{v_r^{-1} p_r[X]}{r} z^r\right) = \exp\left(\sum_{r\geq 1} \frac{p_r\left[X\frac{1-t}{1-q}\right]}{r} z^r\right) = \sigma_z\left[X\frac{1-t}{1-q}\right] \tag{1.3.0.7}$$

Hence, from (1.3.0.4) we find $T(X|Y) = \sigma_1\left[XY\frac{1-t}{1-q}\right]$. ⌟

1.4 Differential operator

We follow [10, I, §5, Exer 3; III, §5, Exer 8]. For each symmetric function $f \in \Lambda$, let $f^{\perp} : \Lambda \to \Lambda$ be the adjoint of multiplication by f, i.e.

$$\langle f^{\perp}, u, v\rangle = \langle u, fv\rangle$$

for all $u, v \in \Lambda$. Then $f \mapsto f^{\perp} : \Lambda \to \mathrm{End}(\Lambda)$ is a ring homomorphism.

First, we will go and describe $p_n^{\perp}$ here. We have $\langle p_n^{\perp} p_\lambda, p_\mu\rangle = \langle p_\lambda, p_n p_\mu\rangle$, which is zero if $\lambda \neq \mu \cup (n)$, and is equal to $z_\lambda v_\lambda$ if $\lambda = \mu \cup (n)$. It follows that $p_n^{\perp} p_\lambda = z_\lambda v_\lambda z_\mu^{-1} v_\mu^{-1} p_\mu$ if n is a part of λ, and μ is the partition obtained by removing one part of n from λ. We know that from definition of z_λ and $\lambda = (n) \cup \mu$, we have $z_\lambda z_\mu^{-1} = n \cdot m_n(\lambda)$ where $m_n(\lambda)$ is the multiplicity of n as part of λ. We also have $v_\lambda v_\mu^{-1} = v_n$. Therefore

$$p_n^{\perp} = n v_n \partial / \partial p_n \tag{1.4.0.1}$$

acting on symmetric functions expressed as polynomials in the p's. In particular, each $p_n^{\perp}$ is a derivation of Λ, i.e. $p_n^{\perp}(fg) = p_n^{\perp}(f)g + f p_n^{\perp}(g)$.

Next, we describe $q_n^{\perp}$. Recall $\Delta : \Lambda_F \to \Lambda_F$ is the comultiplication $\Delta f[X] = f[X+Y]$. From (1.3.0.6), one can show that $\Delta q_n = \sum_{r+s=n} q_r \otimes q_s$. Hence, from (1.3.0.1), we have

$$\langle q_n, gh\rangle = \langle \Delta q_n, g \otimes h\rangle = \sum_{r+s=n} \langle q_r, g\rangle \langle q_s, h\rangle.$$

By taking $g = q_m$, we find that for $m \geq 1$ then

$$\langle q_m^{\perp} q_n, h\rangle = \langle q_n, q_m h\rangle = \sum_{r+s=n} \langle q_r, q_m\rangle \langle q_s, h\rangle = \langle q_m, q_m\rangle \langle q_{n-m}, h\rangle$$

where the last equality comes from (1.3.0.6). Also from (1.3.0.6) and (1.3.0.2), we find

$$\langle q_m, q_m\rangle = \sum_{\lambda \vdash m} \left(\frac{1}{z_\lambda v_\lambda}\right)^2 \langle p_\lambda, p_\lambda\rangle = \sum_{\lambda \vdash m} z_\lambda^{-1} v_\lambda^{-1} = [z^m] \exp\left(\sum_{r \geq 1} \frac{v_r^{-1} p_r[1]}{r} z^r\right).$$

This implies

$$q_m^{\perp} q_n = \begin{cases} \sum_{\lambda \vdash} z_\lambda^{-1} v_\lambda^{-1} q_{n-m} & m \geq 1 \\ q_n & m = 0. \end{cases} \tag{1.4.0.2}$$

Note that when $v_\lambda = p_\lambda\left[\frac{1-q}{1-t}\right]$, from (1.3.0.7) we have

$$\sum_{\lambda \vdash m} z_\lambda^{-1} v_\lambda^{-1} = [z^m] \sigma_z \left[\frac{1-t}{1-q}\right] = [z^m] \prod_{k=0}^{\infty} \frac{1 - tq^k z}{1 - q^k z}.$$

When $v_\lambda = p_\lambda\left[\frac{1}{1-t}\right]$, we can find that

$$\sum_{\lambda \vdash m} z_\lambda^{-1} v_\lambda^{-1} = [z^m] \sigma_z [1-t] = [z^m] \frac{1 - tz}{1 - z} = 1 - t$$

for $m \geq 1$.

Here we include a proposition taken from [10, I, §5, Exer 25].

Proposition 1.4.0.1. We have

$$\Delta f = \sum_{\mu} u_{\mu}^{\perp} f \otimes v_{\mu}$$

whenever $(u_\mu), (v_\mu)$ are dual bases of Λ.

Due to self-duality of comultiplication $\langle \Delta f, g \otimes h\rangle = \langle f, gh\rangle$ so if $f \in \Lambda$, $\Delta f = \sum a_i \otimes b_i$ then

$$f^{\perp}(gh) = \sum_i a_i^{\perp}(g)_i^{\perp}(h)$$

1.5 Vertex operator

Due to notation convenience, we denote $\pi_n : \Lambda \to \Lambda$ to be operators defined as: if $n \leq -1$, π_n is multiplication by p_{-n}; if $n \geq 1$ then $\pi_n = p_n^{\perp} = nv_n \partial/\partial p_n$; and π_0 is the identity. Then from (1.4.0.1), we have

$$[\pi_m, \pi_n] = mv_{|m|}\delta_{m+n,0}\pi_0. \tag{1.5.0.1}$$

for all $m, n \in \mathbb{Z}$. We have the following proposition:

Proposition 1.5.0.1. Consider $\Gamma_{\pm}(z) = \exp\left(\sum_{n\geq 1} z_n \pi_{\pm n}\right)$ where $z = (z_1, z_2, \ldots)$. We have

$$\Gamma_+(z)\Gamma_-(w) = \exp\left(\sum_{n\geq 1} nv_n z_n w_n\right)\Gamma_-(w)\Gamma_+(z) \tag{1.5.0.2}$$

Proof. Define $E_{\pm}(z) := \sum_{n\geq 1} z_n \pi_{\pm n}$ and $F(z, w) = \sum_{n\geq 1} nv_n z_n w_n$ we must show that

$$e^{E_+(z)}e^{E_-(w)} = e^{F(z,w)}e^{E_-(w)}e^{E_+(z)}$$

From (1.5.0.1), we have

$$\begin{aligned}
E_+(z)E_-(w) &= \sum_{m,n\geq 1} z_n w_m \pi_n \pi_{-m}, \\
&= \sum_{n,m\geq 1} z_n w_m (\pi_{-m}\pi_n + nv_{|n|}\delta_{m,n}\pi_0), \\
&= E_-(w)E_+(z) + F(z, w)
\end{aligned}$$

By induction on $k \geq 1$, we have

$$E_+^k(z)E_-(w) = E_-(w)E_+^k(z) + kF(z, w)E_+^{k-1}(z).$$

By induction on $\ell \geq 1$, we have

$$E_+^k(z)E_-^{\ell}(w) = \sum_{i=0}^{\min\{k,\ell\}} i!\binom{k}{i}\binom{\ell}{i} F^i(z, w)E_-^{\ell-i}(w)E_+^{k-i}(z).$$

Since for $k = 0$ or $\ell = 0$ the above identity holds so we can extend the range of k, ℓ to all nonnegative integers. Dividing both sides by $k!\ell!$ and summing over k, ℓ, we obtain the desired identity. $\square$

When $z_n = \frac{1}{nv_n} z^n$, we have

$$\Gamma_+(z)\Gamma_-(w) = \exp\left(\sum_{n\geq 1} \frac{(zw)^n}{nv_n}\right) \Gamma_-(w)\Gamma_+(z). \tag{1.5.0.3}$$

Example 1.5.0.2. When $v_n = \frac{1-q^n}{1-t^n} = p_n\left[\frac{1-q}{1-t}\right]$ then from eq. (1.3.0.7) and (1.5.0.3) we have

$$\Gamma_+(z)\Gamma_-(w) = \sigma_1\left[zx\frac{1-t}{1-q}\right]\Gamma_-(w)\Gamma_+(z) = \prod_{k=0}^{\infty} \frac{1-tq^k zw}{1-q^k zw}\Gamma_-(w)\Gamma_+(z).$$

When $v_n = \frac{1}{1-t^n} = p_r\left[\frac{1}{1-t}\right]$ then similarly, we find

$$\Gamma_+(z)\Gamma_-(w) = \frac{1-tzw}{1-zx}\Gamma_-(w)\Gamma_+(z).$$

$\lrcorner$

We define the vertex operator on Λ as follows

$$B(z) = \exp\left(\sum_{n\geq 1} \frac{1}{nv_n}\pi_{-n} z^n\right) \exp\left(-\sum_{n\geq 1} \frac{1}{nv_n}\pi_n z^{-n}\right) = \sum_{n\in\mathbb{Z}} B_n z^{-n} \tag{1.5.0.4}$$

Define the normal ordering product as:

$$: B(z)B(w) := \exp\left\{\sum_{n\geq 1} \frac{1}{nv_n}\pi_{-n}(z^n + w^m)\right\} \exp\left\{-\sum_{n\geq 1} \frac{1}{nv_n}\pi_n(z^{-n} + w^{-n})\right\}. \tag{1.5.0.5}$$

Applying Proposition 1.5.0.1, we have

$$B(z)B(w) =: B(z)B(w) : \exp\left(-\sum_{n\geq 1} \frac{(w/z)^n}{nv_n}\right) \tag{1.5.0.6}$$

Since $\pi_n \cdot 1 = p_n^{\perp} \cdot 1 = 0$ so $B_{-n} \cdot 1 = \exp\left(\sum_{n\geq 1} \frac{1}{nv_n}\pi_{-n} z^n\right) \cdot 1 = q_n$ according to (1.3.0.3). We have

$$\begin{aligned}
B_{-\lambda} \cdot 1 &= B_{-\lambda_1} B_{-\lambda_2} \cdots B_{-\lambda_k} \cdot 1, \\
&= [z^\lambda] B(z_1) \cdots B(z_k) \cdot 1, \\
&= [z^\lambda] : B(z_1) \cdots B(z_k) : \prod_{1\leq i<j\leq k} \exp\left(-\sum_{n\geq 1} \frac{(z_j/z_i)^n}{nv_n}\right) \cdot 1, \\
&= [z^\lambda] \prod_{1\leq i<j\leq k} \exp\left(-\sum_{n\geq 1} \frac{(z_j/z_i)^n}{nv_n}\right) \exp\left(\sum_{n\geq 1} \frac{p_n[XZ_k]}{nv_n}\right).
\end{aligned}$$

Example 1.5.0.3. In particular, when $v_n = 1$ then we have

$$\prod_{1\le i<j\le k} \exp\left(-\sum_{n\ge 1} \frac{(z_j/z_i)^n}{nv_n}\right) = \prod_{i<j} \sigma_1[-z_j/z_i] = \prod_{1\le i<j\le k} \left(1 - \frac{z_j}{z_i}\right)$$

When $v_n = \frac{1}{1-t^n}$ then

$$\prod_{1\le i<j\le k} \exp\left(-\sum_{n\ge 1} \frac{(z_j/z_i)^n}{nv_n}\right) = \prod_{1\le i<j\le k} \sigma_1\left[-\frac{z_j}{z_i}(1-t)\right] = \prod_{1\le i<j\le k} \frac{1 - z_j/z_i}{1 - tz_j/z_i}.$$

When $v_n = \frac{1-q^n}{1-t^n}$ then

$$\prod_{1\le i<j\le k} \exp\left(-\sum_{n\ge 1} \frac{(z_j/z_i)^n}{nv_n}\right) = \prod_{i<j} \sigma_1\left[-\frac{z_j}{z_i}\frac{1-t}{1-q}\right] = \prod_{i<j}\prod_{k=0}^{\infty} \frac{1 - q^k z_j/z_i}{1 - tq^k z_j/z_i}$$

Since $q_\lambda[X] = [z^\lambda]\exp\left(\sum_{r\ge 1}\frac{p_r[XZ_k]}{rv_r}\right)$ so this leads to the abbreviations in the literatures:

$$s_\lambda = \prod_{i<j}(1 - R_{i,j})q_\lambda$$

when $v_n = 1$ and

$$Q(X;t) = \prod_{i<j} \frac{1 - R_{i,j}}{1 - tR_{i,j}} q_\lambda$$

when $v_n = \frac{1}{1-t^n}$ and

$$B_{-\lambda}\cdot 1 = \prod_{i<j}\prod_{k=0}^{\infty} \frac{1 - q^k R_{i,j}}{1 - tq^k R_{i,j}} q_\lambda$$

when $v_n = \frac{1-q^n}{1-t^n}$. Here $R_{i,j}$ is the raising operator where $R_{i,j}\lambda = (\lambda_1, \dots, \lambda_i + 1, \dots, \lambda_j - 1, \cdots)$ and $Rq_\lambda = q_{Rq_\lambda}$ where R is some product of raising operators $R_{i,j}$. ⌟

1.6 Dual bases of Λ

Our goal for this section is to give a recursive construction (or if possible, an explicit formula) for the family of functions Schur $S_\lambda[X]$, Hall-Littlewood $P_\lambda[X;t]$, Macdonald $P_\lambda[X;q,t]$.

For $Z_k = z_1 + \cdots + z_k$ and λ partition of length at most k. Therefore, from section 1.5, when $v_n = \frac{1-q^n}{1-t^n}$, we have

$$\begin{aligned} B_{-\lambda}\cdot 1 &= [z^\lambda] \prod_{1\le i<j\le k} \exp\left(-\sum_{n\ge 1}\frac{(z_j/z_i)^n}{nv_n}\right)\exp\left(\sum_{n\ge 1}\frac{p_n[XZ_k]}{nv_n}\right), \\ &= [z^\lambda]\prod_{i<j}\prod_{k=0}^{\infty}\frac{1 - q^k z_j/z_i}{1 - tq^k z_j/z_i}\sigma_1\left[XZ_k\frac{1-t}{1-q}\right]. \end{aligned}$$

We will use this to find a recursive formula for $B_{-\lambda} \cdot 1 = B_\lambda[X;q,t]$. To make it easier for us where we try to expand the right-hand-side, we use the substitution $X \mapsto X\frac{1-q}{1-t}$ so that

$$B_\lambda\left[\frac{1-q}{1-t}X;q,t\right] = [z^\lambda]\prod_{i<j}\prod_{k=0}^{\infty}\frac{1-q^k z_j/z_i}{1-tq^k z_j/z_i}\sigma_1\left[XZ_k\right]$$

Example 1.6.0.1 (Schur polynomial)**.** When $q = t$, from example 1.5.0.3, we have

$$B_\lambda[X] = [z^\lambda]\prod_{1\leq i<j\leq k}\left(1-\frac{z_j}{z_i}\right)\sigma_1\left[XZ_k\right].$$

This can be rewritten as

$$B_\lambda[X] = \sigma_1[XZ_k]\Delta(Z_k)|_{z^{\lambda+\rho_k}},$$

where $\rho_k = (k-1, k-2, \ldots, 0)$, $\Delta(Z_k) = \prod_{1\leq i<j\leq k}(z_i - z_j)$ is the Vandemonde determinant with respect to the variables $z_1, \ldots, z_k$. This follows

$$\sigma_1[XZ_k]\Delta(Z_k) = \sum_{|\mu|=k} B_\mu[X]z^{\mu+\rho_k}$$

Since the left-hand-side is an alternating function of Z_k, so the coefficient of $\sigma(z^{\mu+\rho_k})$ must be $(-1)^\sigma B_\mu[X]$, implying

$$\sigma_1[XZ_k]\Delta(Z_k) = \sum_\mu B_\mu[X]\Delta_\mu[Z_k],$$

where $\Delta_\mu = \sum_{w\in S_n}(-1)^w w(x^{\mu+\rho_k}) = \det(x_i^{\rho_j+n-j})$. Note that $\sigma_1[XY] = T(X|Y)$ from (1.3.0.7) and we must have $T(X|Y) = \sum_\lambda B_\lambda[X]B_\mu[Y]$ since B_λ are dual basis with respect to the corresponding scalar product. This follows the determinant formula for the Schur function:

$$s_\lambda[X_n] = \frac{\Delta_\lambda[X_n]}{\Delta[X_n]} = \frac{\det(x_i^{\lambda_j+n-j})}{\det(x_i^{n-j})}.$$

⌟

Example 1.6.0.2 (Hall-Littlewood polynomial)**.** When $q = 0$, from example 1.5.0.3, we have

$$B_\lambda[X;t] = \prod_{1\leq i<j\leq k}\frac{1-z_j/z_i}{1-tz_j/z_i}\sigma_1[XZ_k]\Bigg|_{z^\lambda}.$$

We follow [2] to obtain a recursive definition of the Hall-Littlewood polynomials. From the above, by considering in $n \geq k$ variables $X_n = x_1 + \cdots + x_n$, we have

$$LHS = \sigma_1\left[X_n z_1\right]\prod_{1<j\leq k}\frac{1-z_j/z_1}{1-tz_j/z_1}\Bigg|_{z_1^{\lambda_1}}\cdot\prod_{i=2}^{k}\sigma_1[X_n z_i]\prod_{2\leq i<j\leq k}\frac{1-z_j/z_i}{1-tz_j/z_i}\Bigg|_{z_2^{\lambda_2}\cdots z_k^{\lambda_k}}.$$

Setting
$$\prod_{1<j\le k}\frac{1-y_j/y_1}{1-ty_j/y_1}=\sum_{m\ge 0}c_m(t)y_1^{-m},$$
and using the partial fraction expansion
$$\sigma_1[X_ny]=\prod_{i=1}^{n}\frac{1}{1-x_iy}\sum_{s=1}^{n}\frac{x_s^n}{\prod_{i\neq s}(x_s-x_i)}\frac{1}{1-x_sy},$$
we get
$$\begin{aligned}\sigma_1[X_nz_1]\prod_{1<j\le k}\frac{1-z_j/z_1}{1-tz_j/z_1}\bigg|_{z_1^{\lambda_1}} &= \sum_{m\ge 0}c_m(t)\sigma_1[Xz_1]\bigg|_{z_1^{\lambda_1+m}},\\ &=\sum_{m\ge 0}c_m(t)\sum_{s=1}^{n}\frac{x_s^{\lambda_1+n-1+m}}{\prod_{i\neq s}(x_s-x_i)},\\ &=\sum_{s=1}^{n}\frac{x_s^{\lambda_1+n-1}}{\prod_{i\neq s}(x_s-x_i)}\sum_{m\ge 0}c_m(t)x_s^m,\\ &=\sum_{s=1}^{n}\frac{x_s^{\lambda_1+n-1}}{\prod_{i\neq s}(x_s-x_i)}\prod_{2\le i\le n}\frac{1-x_sz_j}{1-tx_sz_j}.\end{aligned}$$
Note that the effect of factor $\prod_{2\le i\le n}\frac{1-x_sz_j}{1-tx_sz_j}$ on $\prod_{j=2}^{n}\sigma_1[X_nz_j]$ will be to replace x_s by tx_s. We consider this replacement as linear operator on symmetric polynomials an denote it by T_{t,x_s}. With this notation, the resulting formula may be written as
$$B_\lambda[X_n;t]:=P_\lambda[X_n;t]=\sum_{s=1}^{n}\frac{x_s^{\lambda_1+n-1}}{\prod_{i\neq s}(x_s-x_i)}T_sH_{\lambda_2,\dots,\lambda_n}[X_n;t].$$

On the other note, we wonder if one can derive an explicit formula for Hall-Littlewood polynomial as in [10, p. 208] from the vertex operator defined, i.e. following the same route as we did for Schur polynomial in previous example. ⌟

Example 1.6.0.3 (Macdonald's polynomial)**.** We have not figured out how to construct Macdonald's polynomial from vertex operator given in previous section that is similar to what we did in this section to Schur polynomial and Hall-Littlewood polynomial. However, here we will sketch Macdonald's construction [10, p. 321], which includes the appearance of linear operator T_{t,x_s} that is indicated in Hall-Littlewood's polynomial construction in previous example.

The main idea is that in order to find dual basis with respect to the scalar product
$$\langle p_\lambda,p_\mu\rangle=z_\lambda p_\lambda\left[\frac{1-q}{1-t}\right]\delta_{\lambda\mu}.$$
on Λ, we construct a self-adjoint operator E with respect to that scalar product. And in order to do that, we restrict to finite number of variables Λ_n and to scalar

product $\langle , \rangle_n$ on Λ_n. From here we try to find a family of self-adjoint operators D_n^1 with respect to the scalar product $\langle , \rangle_n$ such that upon modification to get E_n, we can take the limit $n \to \infty$ for these operators to obtain E. In other words, we would like E_n to be able to commute with the restriction map $\rho_{n,n-1} : \Lambda_n \to \Lambda_{n-1}$ by assigning $x_n \mapsto 0$.

Finally, we will define E in this example: Consider symmetric functions over field F of characteristic 0. For $u \in F$ and $1 \leq i \leq n$ we define *shift operator* $T_{u,x_i} : \Lambda_{n,F} \to \Lambda_{n,F}$ by

$$T_{u,x_i} f(x_1, \ldots, x_n) = f(x_1, \ldots, ux_i, \ldots, x_n)$$

for any polynomial $f \in F[x_1, \ldots, x_n]$. Next, let X be another indeterminate and define

$$D_n[X; q, t] = a_\delta(x)^{-1} \sum_{w \in S_n} \varepsilon(w) x^{w\delta} \prod_{i=1}^{n} \left(1 + X t^{(w\delta)_i} T_{q,x_i}\right) = \sum_{r=0}^{n} D_n^r X^r.$$

where $\delta = (n-1, \ldots, 1, 0)$, $a_\delta(x)$ is the Vandemonde determinant, $\varepsilon(w) = \pm 1$ is the sign of $w \in S_n$, and $(w\delta)_i$ is the ith component of $w\delta$. We obtain family of operators D_n^r for $0 \leq r \leq n$. These are the self-adjoint operators with respect to the scalar product $\langle , \rangle_n$, i.e.

$$\langle D_n^r f, g \rangle_n = \langle f, D_n^r g \rangle_n$$

for $f, g \in \Lambda_{n,F}, 0 \leq r \leq n$. Considering our goal, it is sufficient to choose one of these operators for each n, and here D_n^1 is chosen for simplicity in computation. Finally, upon modification, we obtain

$$E_n = t^{-n} D_n^1 - \sum_{i=1}^{n} t^{-i},$$

family of self-adjoint operators with respect to $\langle , \rangle_n$, and commuting with $\rho_{n,n-1}$. With this, we can take the limit $n \to \infty$ to obtain $E : \Lambda \to \Lambda$. Its eigenfunctions are precisely the Macdonald functions $P_\lambda[X; q, t]$. ⌟

1.7 Hopf algebra structure of Λ

We point out that Λ has a structure of a self-dual cocommutative Hopf algebra over $\mathbb{Z}$. We will write down the definitions in details and the maps that give Λ this structure without going through verification. One can find the proof in [11], for example.

Instead, in this section, we will try to explain why plethystic notation helps to simplify our calculation in symmetric functions theory and we do this by looking at the algebraic structure of Λ. The point here is that each of the defined structured map of Λ has a plethystic analog (which can be seen as basis-free definition of the maps) that makes many of the commutative diagrams turn into simple-looking identities, many of which agrees with our view of plethystic notation indicated in previous section.

Proof. Before this, we identify elements of $\Lambda \otimes \Lambda$ with functions of two set of alphabets X, Y, symmetric in each set *separately*: thus, $f \otimes g$ corresponds to $f[X]g[Y]$. We will see the usefulness of such identification shortly.

We start with the general notation $(A, m, e, \Delta, \varepsilon, S)$ for Hopf algebra:

1. (A, m, e) is an associative k-algebra, equipped with k-linear multiplication $m : A \otimes A \to A$ and k-linear unit $e : k \to A$ satisfying the following commutative diagrams:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mathrm{id}\otimes m"] \arrow[d, "m\otimes\mathrm{id}"] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd} \qquad \begin{tikzcd} A \otimes k \arrow[d, "\mathrm{id}\otimes e"] & A \arrow[l, "a\mapsto a\otimes 1"'] \arrow[r, "a\otimes 1\to a"] \arrow[d, "\mathrm{id}"] & k \otimes A \arrow[d, "e\otimes\mathrm{id}"] \\ A \otimes A \arrow[r, "m"] & A & A \otimes A \arrow[l, "m"'] \end{tikzcd} \tag{1.7.0.1}$$

   When $A = \Lambda$, multilplication is the usual product and unit $e(1) = 1$ then extends linearly. Commutative diagrams follows from this.

   Via plethystic notation, multiplication $m : \Lambda \otimes \Lambda \to \Lambda$ is viewed as $f \otimes g = f[X]g[Y] \mapsto f[X]g[Y]$. Note that $f[X]g[Y]$ on RHS is indeed in $\Lambda = \Lambda(X, Y)$ as it is symmetric in the set of variables (X, Y). Thus, in this notation, we denote $f[X]g[Y]$ to indicate that it can be element of $\Lambda \otimes \Lambda$ or element of Λ.

   Unit $e : \mathbb{Z} \to \Lambda$ is viewed as $1 \mapsto 1[X] \equiv 1$ and extends linearly.

   The commutative diagram then indicate that $(f[X]g[Y])h[Z] = f[X](g[Y]h[Z])$ and $f[X]1[Y] = f[X] = 1[X]f[Y]$.

2. (A, Δ, ε) is a co-associative coalgebra (A, Δ, ε), which consists of k-linear comultiplication $\Delta : A \to A \otimes A$ and k-linear counit map $\varepsilon : A \to k$ satisfying the following commutative diagrams:

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"] & A \otimes A \arrow[d, "\mathrm{id}\otimes\Delta"] \\ A \otimes A \arrow[r, "\Delta\otimes\mathrm{id}"] & A \otimes A \otimes A \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \arrow[d, "\mathrm{id}\otimes\varepsilon"] & A \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "\mathrm{id}"] & A \otimes A \arrow[d, "\varepsilon\otimes\mathrm{id}"] \\ A \otimes k \arrow[r] & A & k \otimes A \arrow[l] \end{tikzcd} \tag{1.7.0.2}$$

When $A = \Lambda$ then $\Delta(p_r) = p_r \otimes 1 + 1 \otimes p_r$ and $\varepsilon(p_r) = r$ and commutative diagrams follow.

Via plethystic notation, comultiplication $\Delta : \Lambda \to \Lambda \otimes \Lambda$ is seen as $f[X] \mapsto f[X+Y]$ and counit $\varepsilon : \Lambda \to \mathbb{Z}$ as $f[X] \mapsto f[0]$. Here, $f[X+Y]$ is viewed as an element of $\Lambda \otimes \Lambda$. The commutative diagrams then indicate that $f[(X+Y)+Z] = f[X+(Y+Z)]$ and that $f[X] \equiv f[Y]$.

3. For $(A, m, e, \Delta, \varepsilon)$ to be called *bialgebra* over k, we need a compatible condition for the previous two structures:

   The maps Δ and ε are morphisms for the algebra structure (A, m, e). Or equivalently, the maps m and e are morphisms for the coalgebra structure (A, Δ, ε). Or equivalently, the following diagrams commutes:

$$
\begin{tikzcd}
 & A \otimes A \arrow[dl, "\Delta\otimes\Delta"'] \arrow[ddr, "m"] & \\
A \otimes A \otimes A \otimes A \arrow[dd, "\mathrm{id}\otimes T\otimes\mathrm{id}"'] & & \\
 & & A \arrow[ddl, "\Delta"] \\
A \otimes A \otimes A \otimes A \arrow[dr, "m\otimes m"'] & & \\
 & A \otimes A &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \otimes A \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "m"'] & \mathbf{k} \otimes \mathbf{k} \arrow[d, "m"] \\
A \arrow[r, "\varepsilon"'] & \mathbf{k}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{k} \arrow[r, "e"] \arrow[d, "\Delta"'] & A \arrow[d, "\Delta"] \\
\mathbf{k} \otimes \mathbf{k} \arrow[r, "e\otimes e"'] & A \otimes A
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbf{k} \arrow[rr, "\mathrm{id}"] \arrow[dr, "e"'] & & \mathbf{k} \\
 & A \arrow[ur, "\varepsilon"'] &
\end{tikzcd}
\tag{1.7.0.3}
$$

   Here $T : A \otimes A \to A \otimes A$ is the *twist map* defined as $a \otimes b \mapsto b \otimes a$. In case of $A = \Lambda$ and in plethystic notation, this is $f[X]g[Y] \mapsto g[X]f[Y]$.

   One can show the diagram commutes by checking it with p_λ.

   Via plethystic notation, some commutative diagrams just turn into trivial-looking identities. For example, we have $1[X+Y] \equiv 1[X]1[Y]$ or $1[0] = 1$.

4. k-coalgebra (A, Δ, ε) is *commutative* if this diagram commutes

$$\begin{tikzcd}
A \otimes A \arrow[rr, "T"] & & A \otimes A \\
& A \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

   In plethystic notation, the diagram indicates $f[X+Y] = f[Y+X]$.

5. In order to make a bialgebra into a Hopf algebra, we need an *antipode* $S \in \mathrm{End}(A)$ which must be a two-sided inverse under *convolution* $\star$ on $\mathrm{End}(A)$. [For coalgebra C and algebra A, we endow k-module $\mathrm{Hom}(C, A)$ with associative algebra structure by defining the product $f \star g$ to be the composite

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{m} A$$

   ] Equivalently, the following diagram commutes

$$\begin{tikzcd}
& A \otimes A \arrow[rr, "S \otimes \mathrm{id}_A"] & & A \otimes A \arrow[dr, "m"] & \\
A \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & \mathbf{k} \arrow[rr, "e"] & & A \\
& A \otimes A \arrow[rr, "\mathrm{id}_A \otimes S"'] & & A \otimes A \arrow[ur, "m"'] &
\end{tikzcd} \tag{1.7.0.4}$$

   Here our antipode is $S(p_\lambda) = (-1)^{l(\lambda)} p_\lambda$ and the commutative diagram follows. In plethystic notation S sends $f[X] \mapsto f[-X]$ and the diagram then indicates

$$f[X] \xmapsto{\Delta} f[X+Y] \xmapsto{S \otimes \mathrm{id}} f[-X+Y] \xmapsto{m} f[-X+X] = f[0] = f[0]1[X]$$

   From $f[-X+Y] \xmapsto{m} f[-X+X]$, one sees the meaning of plethystic minus in $-X+Y$ as set exclusion when X, Y viewed as sets.

6. Λ is *self-dual* with respect to the inner product on $\Lambda \otimes \Lambda$ defined as $\langle f_1 \otimes g_1, f_2 \otimes g_2 \rangle = \langle f_1, f_2 \rangle \langle g_1, g_2 \rangle$ where the inner product on the RHS is the Hall inner product on Λ. That is, we have $\langle \Delta f, g \otimes h \rangle = \langle f, gh \rangle$.

   In plethystic notation, this is $\langle f[X+Y], g[X]h[Y] \rangle = \langle f[X], g[X]h[X] \rangle$. It seems that even in plethystic notation, this property of Λ is not as clear as we see for plethystic notation in other properties.

□

By using plethystic notation, we use the same notation and $f[X+Y]$ to mean them as elements of Λ and $\Lambda \otimes \Lambda$. This leads to a simple description going between any $\Lambda^{\otimes k}$.

1.8 Another scalar product on Λ_n

In this section, we define a scalar product on Λ_n, symmetric polynomials of n variables, and show its relation with previous scalar product on Λ that we encountered.

First, we consider elements in $\mathbb{Q}[x_1, \ldots, x_n]$ as complex-valued function on $\mathbb{R}^n = \{(t_1, \ldots, t_n)\}$. This can be done by considering x_i as the function $x_k(t_1, \ldots, t_n) = \exp(2\pi i t_k)$ then extend it by multiplicativity and linearity. Note that $\overline{x_k} = x_k^{-1}$. These functions are well defined on the torus $T = \mathbb{R}^n/\mathbb{Z}^n$, and

$$\int_T x^\lambda dt = \begin{cases} 1, & \lambda = 0, \\ 0, & \lambda \in \mathbb{Z}, \lambda \neq 0. \end{cases}$$

This follows that for any Laurent polynomial f over $\mathbb{C}$ then $\int_T f dt$ is precisely the coefficient of 1 in f.

The monomial polynomial $m_\lambda(x_1, \ldots, x_n)$ can be uniquely characterised by two following conditions:

(a) The triangularity condition: $m_\lambda = \mathbf{x}^\lambda + \text{lower terms}$. Here "lower terms" means a linear combination of $^\mu$ with $\mu \prec \lambda$. Here $\prec$ is the lexicographic ordering on partitions: $\mu \prec \lambda$ if $\sum \mu_i = \lambda_i$ and the smallest k such that $\mu_k \neq \lambda_k$ will have $\mu_k < \lambda_k$.

(b) The orthogonality condition:

$$\int_T m_\lambda \overline{m_\mu} dt = 0 \text{ for } \lambda \neq \mu.$$

Similarly, the Schur polynomial $s_\lambda(x_1, \ldots, x_n)$ can also be uniquely characterised by two conditions:

(a) The triangularity condition:

$$s_\lambda = m_\lambda + \sum_{\mu < \lambda} K_{\lambda\mu} m_\mu$$

for suitable coefficients $K_{\lambda\mu}$. Here $<$ is the *dominance partial ordering* on partitions: $\lambda > \mu$ means $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all $i \geq 1$.

(b) The orthogonality condition:

$$\int_T \prod_{i<j} |x_i - x_j| s_\lambda \overline{s_\mu} dt = 0 \text{ for } \lambda \neq \mu.$$

Note that $|x_i - x_j| = x_i^{-1} - x_j^{-1}$ when viewing x_i, x_j as functions $\mathbb{R}^n \to \mathbb{C}$.

This follows that the Schur function is the orthonormal basis of Λ_n with respect to the following scalar product:

$$\langle f,g\rangle'_n = \frac{1}{n!}\int_T \prod_{i<j} f\delta\overline{g}\overline{\delta}dt = \frac{1}{n!}[fg\Delta]_0, \tag{1.8.0.1}$$

where $\delta = \prod_{i<j}(x_i - x_j)$ and $\Delta = \delta\bar{\delta}$; $[f]_0$ denotes the constant coefficient of Laurent polynomial f. This is precisely the scalar product that is given for Schur function in section 1.3, where we restrict to n variables.

One may wonder if there is a similar scalar product for Macdonald's polynomials $P_\lambda(x_1,\dots,x_n;q,t)$. Indeed, Macdonald [10, p. 368] introduced the following scalar product on $\mathbb{C}(q,t)[x_1^\pm,\cdots,x_n^\pm]$:

$$\langle f,g\rangle_{n;q,t} = \frac{1}{n!}\int_T f(z)\overline{g(z)}\Delta(z;q,t)dz = \frac{1}{n!}[f\overline{g}\Delta]_0,$$

where

$$\Delta = \Delta(x;q,t) = \prod_{i\neq j}\frac{(x_i x_j^{-1};q)_\infty}{(tx_i x_j^{-1};q)_\infty}$$

where $(a;q)_\infty = \prod_{r=0}^\infty(1-aq^r)$. Here Δ should be considered as Laurent series in q,t with coefficients from $\mathbb{C}[x_1^\pm,\cdots,x_n^\pm]$ and the scalar product also takes values in the Laurent series. The polynomials $P_\lambda(x;q,t)$ where $x = (x_1,\dots,x_n)$ and $l(\lambda)\leq n$ are pairwise orthogonal with respect to this scalar product. [6]

There is also a question of relation between this scalar product and the previously defined scalar product $\langle,\rangle$ in section 1.3. Macdonald [10, p. 371] showed that upon normalization, we have

$$c_n^{-1}\langle\rho_n f,\rho_n g\rangle_{n;q,t}\to\langle f,g\rangle$$

as $n\to\infty$, where $f,g\in\Lambda_F$, $c_n = \langle 1,1\rangle_{n;q,t}$ and $\rho_n:\Lambda_F\to\Lambda_{n,F}$ is the restriction map to n variables.

We will soon see in chapter 2 that approaching P_λ via this scalar product can be generalized for any root systems as this scalar product appears naturally in representation theory of compact groups.

1.9 Transitions between bases

As our Λ contains so many bases, one may wonder if there is any nice description going from one basis to the other. There are three major descriptions as we know. One is going from Schur function to the monimial symmetric function:

$$s_\lambda = \sum_{\mu<\lambda} K_{\lambda\mu}m_\mu.$$

[6]It seems for the integration over torus to make sense, our field needs to be over $\mathbb{C}$, but one can always take the second definition of $\langle,\rangle_{n;q,t}$ as constant term of $f\overline{g}\Delta$ and work for any field F of characteristic 0.

Recall $\mu < \lambda$ is the dominance partial ordering given by $\sum_{i=1}^{n} \mu_i \leq \sum_{i=1}^{k} \lambda_i$ for all $k \geq 1$. $K_{\lambda\mu}$, called *Kostka numbers*, counts number of tableau of shape λ and weight μ (i.e. number of ways to fill the Young diagram of λ with μ_i number i such that it is strictly increasing down each column and increasing along each row from left to right).

When we take the t-deformation of the Schur function, we obtain the Hall-Littlewood function $P_\lambda(x;t)$. The end result is:

$$s_\lambda(x) = \sum_{\mu<\lambda} K_{\lambda\mu}(t) P_\mu(x;t),$$

where $K_{\lambda\mu}(1) = K_\lambda$ is the Kostka number. $K_{\lambda\mu}(t)$ is called *Kostka-Foulkes polynomials*. It has a very nice combinatorial description that generalises the previous Kostka number. The proof can be read from [1]. It can be shown that there are certain statistic called *charge* defined on tableau T such that

$$K_{\lambda\mu}(t) = \sum_{T} t^{c(T)}$$

summed over all tableau T of shape λ and weight μ. We refer to the description of such charge in [1, 10, **?**].

The third combinatorial description is the *Littlewood-Richardson rule*, which aims to construct the coefficient $c_{\mu\nu}^{\lambda}$ in

$$s_\mu s_\nu = \sum_{\lambda} s_{\mu\nu}^{\lambda} s_\lambda.$$

In particular, $c_{\mu\nu}^{\lambda}$ counts number of tableau T of shape $\lambda - \mu$ and weight ν such that $w(T)$ is a *lattice permutation*. Here we define this terminology: For a tableau T, we derive a *word* or sequence $w(T)$ by reading the symbols in T from right to left in successive rows, starting from top to bottom. If a word arises this way for $\lambda - \mu$, we say w is *compatible* with $\lambda - \mu$. A word $a_1 \cdots a_N$ of symbols $1, \ldots, n$ is a lattice permutation if for $1 \leq r \leq N$ and $1 \leq i \leq n-1$, the number of occurences of the symbol i in $a_1 a_2 \cdots a_r$ is not less than number of occurrences of $i+1$.

Chapter 2

Generalization for any root systems

2.1 Root systems

Let V be finite dimensiona real vector space, endowed with positive-definite bilinear form $\langle u, v \rangle$. For each nonzero $\alpha \in V$, let s_α denote the orthogonal relfection in the hyperplane through the origin perpendicular to α, so that

$$s_\alpha(v) = v - (v, \alpha^\vee)\alpha$$

for $v \in V$, where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$.

A *root system* R in V is finite nonempty set of nonzero vectors (called *roots*) that span V and are such that for each pair $\alpha, \beta \in R$ we have

$$(\alpha^\vee, \beta) \in \mathbb{Z}$$

and

$$s_\alpha(\beta) \in R.$$

Thus, each reflection $s_\alpha(\alpha \in R)$ permutes R, and the group of orthogonal tranformation of V generated by s_α is a finite group W called the *Weyl group* of R. The vectors $\alpha^\vee$ for $\alpha \in R$ form a root system $R^\vee$.

Let R be any (reduced, irreducible) root system and let $v \in V$ such that $(v, \alpha) \neq 0$ for each $\alpha \in R$. Then the set R^+ of roots $\alpha \in R$ such that $(v, \alpha) > 0$ is called *positive roots* in R. Here R depends on the choice of $v \in V$ but it can be shown that any system of positive roots is of the form wR^+ for a unique element $w \in W$.

Root $\alpha \in R^+$ is *simple* if it is not sum of two elements in R^+. We call the reflection s_i of the simple root α_i the *simple reflection*. Denote Δ to be the set of all simple roots.

Here is a proposition stating the main property about simple roots (its proof contains more information about simple roots).

Proposition 2.1.0.1 (More about simple roots)**.** Each element $\alpha \in R$ can be expressed uniquely as integral linear combination of simple roots $\alpha = \sum_i n_i\alpha_i$ where $n_i \in \mathbb{Z}$ and either all $n_i \geq 0$ if $\alpha \in R^+$ or all $n_i \leq 0$ if $\alpha \in R^-$.

Proof. We prove the proposition by proving following claims in order:

(1) *For two roots α, β, if $(\alpha, \beta) > 0$ (i.e. α, β form obtuse angle) then $\alpha - \beta$ is a root. If $(\alpha, \beta) < 0$ then $\alpha + \beta$ is a root.*

Since $(\alpha, \beta^\vee)(\beta, \alpha^\vee) = 4\cos^2\theta$ where θ is the angle between two roots α, β, we have a limited number of cases to consider.

(2) *Each root in R^+ is a nonnegative-integral linear combination of α_i's.* We order the positive roots so that if $(v, \alpha) > (v, \beta) > 0$ then $\alpha > \beta$ (note that we define R^+ with respect to certain $v \in V$). If there exists α that is not $\mathbb{Z}_{\geq 0}$-linear combination of α_i, then by choosing the minimal such α with respect to $>$, we know α is not a simple root, so there exists $\beta_1, \beta_2 \in R^+$ so $\alpha = \beta_1 + \beta_2$. This implies $\beta_1, \beta_2 < \alpha$, meaning β_1, β_2 are $\mathbb{Z}_{\geq 0}$-linear combination of simple roots α_i, a contradiction to α.

(3) *If α, β are simple roots then either $\alpha = \beta$ or $(\alpha, \beta) \leq 0$.* If $\alpha \neq \beta$, we show $\alpha - \beta$ is not a root. Indeed, suppose the contrary, WLOG, suppose $\alpha - \beta$ is positive root, then $\alpha = (\alpha - \beta) + \beta$ is a simple root written as sums of two positive roots, a contradiction. Thus, $\alpha - \beta$ is not a root, implying $(\alpha, \beta) \leq 0$ from (1).

(4) *The simple roots α_i are linearly independent.* Suppose $\sum n_i\alpha_i = 0$ for $n_i \in \mathbb{R}$. By seperating α_i where $n_i > 0$ from those for which $n_i < 0$, we can rewrite this as $\varepsilon = \sum_\alpha n_\alpha \alpha = \sum_\beta t_\beta \beta$ with $n_\alpha, t_\beta > 0$. This implies $\varepsilon = \sum_{\alpha,\beta} n_\alpha t_\beta(\alpha, \beta) \leq 0$ as $(\alpha, \beta) \leq 0$ from (3). This forces $\varepsilon = 0$ so $0 = (v, \varepsilon) = \sum_\alpha n_\alpha(v, \alpha)$ so $n_\alpha = 0$ and similarly, $t_\beta = 0$. Thus, α_i's are linearly independent.

(5) *If α_i simple and $\beta \in R^+$, then either $\alpha_i = \beta$ or $s_i(\beta) \in R^+$.* □

An element $\gamma \in V$ is *regular* if $(\gamma, \alpha) \neq 0$ for all $\alpha \in R$. Such element exists since R is finite.

Consider element $w \in W$, written as product of simple reflections $w = s_{i_1} \cdots s_{i_k}$. We say $s_{i_1} \cdots s_{i_k}$ is a *reduced* expression of w, if w cannot be written as a product of less than k simple reflections, and we write $\ell(w) = k$ to be the *length* of w. We denote $\ell(1) = 0$.

Proposition 2.1.0.2 (More about Weyl groups)**.** Weyl group W is a finite group generated by simple reflections s_i. W acts transitively on sets of simple roots and its behaviour can be understood via its action on R (rather than on the whole vector space V). Furthermore, in the below proof, we describe some property of the simple reflections: how they interact with each other, how they act on R.

Proof. (1) *s_i permutes the positive roots $R^+ \setminus \{\alpha_i\}$. In particular, the simple reflection s_i permutes all simple roots except α_i where $s_i(\alpha_i) = -\alpha_i$.*

From Proposition 2.1.0.1, we can let $\beta = \sum n_i\alpha_i$. Since $\beta \in R^+$ so all $n_i \geq 0$. We have $s_i\beta = \beta - (\beta, \alpha_i^\vee)\alpha_i$ so passing from β to $s_i\beta$, only the coefficient of α_i is modified. Since $\beta \in R^+ \setminus \{\alpha_i\}$ so there exists coefficient $n_j > 0$ for $j \neq i$. This follows $s_i(\beta)$ has at least one postive coefficient (that is not of α_i), implying that it is in $R^+ \setminus \{\alpha_i\}$. It is not hard to show that s_i is permutation of $R^+ \setminus \{\alpha_i\}$ from here.

(2) *If w is any orthogonal endomorphism of V with respect to $(,)$, then $ws_\alpha w^{-1} = s_{w(\alpha)}$.*

(3) (Exchange property) *If $s_1 \cdots s_k(\alpha) \in R^-$ for some simple reflections s_i's and simple root $\alpha \in \Delta$. Then there exists $1 \le j \le k$ such that*

$$s_1 \cdots s_k = s_1 s_2 \cdots \hat{s_j} \cdots s_k s_\alpha$$

where the hat in $\hat{s_j}$ means we omit the element s_j in that product. Since $s_1 \cdots s_k(\alpha) \in R^-$ and $\alpha \in R^+$ so there exists minimal $1 \le j \le k$ such that $s_{j+1} \cdots s_k(\alpha) \in R^+$. Then $s_j s_{j+1} \cdots s_k(\alpha) \in R^-$ due to minimality of j. From (1), we know the only element of R^+ mapped into R^- by s_j is α_j. Therefore,

$$s_{j+1} \cdots s_k(\alpha) = \alpha_j.$$

This follows

$$(s_{j+1} \cdots s_k) s_\alpha (s_{j+1} \cdots s_k)^{-1} = s_j,$$

which is equivalent to what we want to prove.

(4) *If $s_1 \dots, s_k$ is reduced, then $s_1 \cdots s_k(\alpha_k) \in R^-$ for simple reflection α_k where $s_k = s_{\alpha_k}$.* Since $s_k \alpha_k = -\alpha_k$ so $s_1 \cdots s_k(\alpha_k) = -s_1 \cdots s_{k-1}(\alpha_k)$. Since $s_1 \cdots s_k$ is reduced so from previous proposition implies that $s_1 \cdots s_{k-1}(\alpha_k) \in R^+$, as desired.

(5) *If $w = s_1 \cdots s_k$ is a reduced expression, then k is number of positive roots α such that $w(\alpha) \in R^-$. In particular, the set of positive roots sent into negative roots by w is set of element $\beta_j := s_k s_{k-1} \cdots s_{j+1}(\alpha_j)$ for $1 \le j \le k$.* Since $s_k \cdots s_{j+1} s_j$ is reduced so from (4), $\beta_j \in R^+$. Since $s_1 s_2 \cdots s_j$ is reduced so $w(\beta_j) = s_1 \cdots s_j(\alpha_j) \in R^-$, again from (4). Conversely, if there exists positive root $\beta \in R^+$ such that $w\beta \in R^-$. Arguing as in (3), there exists $1 \le j \le k$ such that $s_{j+1} \cdots s_k(\beta) = \alpha_j$, i.e. $\beta = \beta_j$.

(6) *If $\alpha \in R$, there exists $w \in W$ such that $w(\alpha) \in \Delta$ is simple root.* First, assume that $\alpha \in R^+$. Note that from Proposition 2.1.0.1, for each $\alpha \in R^+$, by writting it as $\alpha = \sum_i n_i \alpha_i$, we can define a positive integer $h(\alpha) = \sum_i n_i$, called *height* of α. We induct on $h(\alpha)$. If $h(\alpha) = 1$, meaning it is a simple root then $w = 1$. If $h(\alpha) > 1$ then as $0 < (\alpha, \alpha) = \sum_\beta n_\beta (\alpha, \beta)$, there exists simple root $\beta \in \Delta$ such that $(\alpha, \beta) > 0$, and then with $\alpha' = s_\beta(\alpha)$ we have $h(\alpha') < h(\alpha)$. On the other hand, $\alpha' \in R^+$ since $\alpha \neq \beta$ by Proposition 2.1.0.1[5]. By inductive hypothesis, $w'(\alpha') \in \Delta$ for some $w' \in W$. Then $(w' s_\beta)(\alpha) \in \Delta$.

If on the other hand $\alpha \in R^-$, then $-\alpha \in R^+$ so we find $w' \in W$ so $w(-\alpha) \in \Delta$. Letting $w'(-\alpha) = \beta$, we have $w(\alpha) = \beta$ with $w = s_\beta w'$.

(7) *W is generated by simple reflections.* Let W' to be subgroup of W' generated by the simple reflections. Then all previous claims we made also hold for W' (espescially (6)). It suffices to show W' contains s_α for $\alpha \in R$. Indeed, by (6), $w(\alpha) \in \Delta$ or some $w \in W'$, so $s_{w(\alpha)} \in W'$. Hence, $s_{w(\alpha)}$ and s_α is conjugate in W' by (2), implying $s_\alpha \in W'$. Thus, $W = W'$.

(8) *If $w(R^+) = R^+$ then $w = 1$.* If $w(R^+) = R^+$, then $\ell(w) = 0$ by (5). This follows $w = 1$.

(9) *W is finite.* By Proposition 2.1.0.1[5], we know that W preserves R. On the other hand, given any $w, w' \in W$ such that their action on R is the same, (8) implies that $w = w'$. Thus, since R is finite, W is finite. $\square$

Root lattice Q: abelian group generated by R; elements are integral linear combinations of the roots. $Q^+ \subset Q$ consisting of all sums $\sum m_i \alpha_i$ where the coefficients are nonnegative integers.

Weight lattice P is set of all $\lambda \in V$ such that $(\lambda, \alpha^\vee) \in \mathbb{Z}$ for all $\alpha \in R$. Denote P^+ set of *dominant weights* $\lambda \in P$ such that $(\lambda, \alpha^\vee) \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in R^+$. Denote $P^+ +$ set of *regular dominant weights* $\lambda \in P^{++}$ such that $(\lambda, \alpha^\vee) \in \mathbb{Z}_{>0}$ for all $\alpha \in R^+$. Here *regular* refers to element $\lambda \in V$ such that $(\lambda, \alpha) \neq 0$ for all $\alpha \in R$. [1]

Proposition 2.1.0.3 (More about weight/root lattices**).**

a) An equivalent definition for P is that $(\lambda, \alpha_i^\vee) \in \mathbb{Z}$ for all simple roots α_i. Dominant weights $\lambda \in P^+$ has equivalent definition of $(\lambda, \alpha_i^\vee) \in \mathbb{N}$ for all i.

b) We have $P \supset Q$ but $P^+ \not\supset Q^+$ (unless $n = 1, R = A_1$). The quotient P/Q is a finite group, since both P and Q are lattices of the same rank. Both P and Q are stable under the action of the Weyl group, and W acts trivially on P/Q.

c) Each W-orbit in P contains exactly one dominant weight, i.e. P^+ is a fundamental region of the action of W on P. If we define a partial order on V by $\mu < \lambda \iff \lambda - \mu \in Q^+$ and $\lambda \neq \mu$, then the unique dominant weight corresponding to each W-orbit is maximal with respect to this partial order in this W-orbit. To be more specific, if $\lambda \in P^+$ then $\lambda \geq w(\lambda)$ for all $w \in W$ and if $\lambda \in P^{++}$ then $\lambda > w(\lambda)$ for all $w \in W$ unless $w = 1$.

Proof. (1) *An equivalent definition for P is that $(\lambda, \alpha_i^\vee) \in \mathbb{Z}$ for all simple roots α_i. Dominant weights $\lambda \in P^+$ has equivalent definition of $(\lambda, \alpha_i^\vee) \in \mathbb{Z}_{\geq 0}$ for all i.* Consider $\alpha \in R$. We know that there exists $w \in W$ and index i such that $\alpha = w\alpha_i$. We can then write

$$\alpha^\vee = \frac{2}{(\alpha, \alpha)}\alpha = \frac{2}{(w\alpha_i, w\alpha_i)} w\alpha_i = w\alpha_i^\vee.$$

Since W is generated by s_j and we have

$$s_j(\alpha_i^\vee) = \alpha_i^\vee - (\alpha_j, \alpha_i^\vee)\alpha_j^\vee,$$

where $(\alpha_j, \alpha_i^\vee) \in \mathbb{Z}$. Therefore, $w\alpha_i^\vee = \alpha^\vee$ can be written as integral linear combination of $\alpha_i^\vee$'s. Hence, $(\lambda, \alpha_i^\vee) \in \mathbb{Z}$ for all i would imply $(\lambda, \alpha^\vee) \in \mathbb{Z}$.

(2) *P, Q are stable under W.* For $v \in P, \alpha, \beta \in R$ then

$$(s_\alpha(v), \beta^\vee) = (v, \beta^\vee) - (v, \alpha^\vee)(\alpha, \beta^\vee) \in \mathbb{Z}.$$

[1] To highlight the geometry, the set is called P^{++} the *(open) Weyl chamber*, obtained by taking the connected component of the complement of union of hyperplanes $\{x \in V | (x, \alpha) = 0 \ \forall \alpha \in R\}$. P^+ is then its closure, called *Weyl chamber*.

So P is stable under action of W.

(3) *If $\lambda \in P^+$ then $\lambda \geq w(\lambda)$ for all $w \in W$ and if $\lambda \in P^{++}$ then $\lambda > w(\lambda)$ for all $w \in W$ unless $w = 1$.* Let $\lambda \in P$. For simple root α_i, we have

$$s_i(\lambda) := s_{\alpha_i}(\lambda) = \lambda - (\lambda, \alpha_i^\vee)\alpha_i.$$

so $W(\lambda) \subset \lambda + Q$, where $W(\lambda)$ is the W-orbit of λ. If $\lambda \in P^+$ then $(\lambda, \alpha_i^\vee) \geq 0$ for every i and so $s_i(\lambda) \leq \lambda$; if $\lambda \in P \setminus P^+$ then there exists some i such that $(\lambda, \alpha_i^\vee) < 0$, i.e. some i so $s_i(\lambda) > \lambda$. But W is finite, so for any $\lambda \in P$, $W(\lambda)$ has a maximal element, which must be in P^+. □

2.2 Weyl characters and their scalar product

Let $A = \mathbb{Z}[P]$ be group algebra over $\mathbb{Z}$ of the weight lattice P. [2] Since the group operation in P is addition, we use exponential notation in A, and denote e^λ the element of A corresponding to $\lambda \in P$. These "formal exponentials" e^λ form an $\mathbb{Z}$-basis of A, such $e^\lambda e^\mu = e^{\lambda+\mu}$ and $(e^\lambda)^{-1} = e^{-\lambda}$. In particular, $e^0 = 1$ is the identity of A.

The Weyl group acts on P and therefore also on A: $w(e^\lambda) = e^{w\lambda}$ for $\lambda \in P$ and $w \in W$. We denote A^W subalgebra of W-invariant elements in A.

Since each W-orbit in P meets P^+ exactly once, it follows that the *orbit-sums*

$$m_\lambda = \sum_{\mu \in W\lambda} e^\mu \tag{2.2.0.1}$$

where $\lambda \in P^+$ and $W\lambda$ is the W-orbit of λ, form an F-basis of A^W.

Another $\mathbb{Z}$-basis of A^W is obtained as follows. Let

$$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha \tag{2.2.0.2}$$

and let

$$\delta = \prod_{\alpha \in R^+} (e^{\alpha/2} - e^{-\alpha/2}) = e^\rho \prod_{\alpha \in R^+} (1 - e^{-\alpha}) = e^{-\rho} \prod_{\alpha \in R^+} (e^\alpha - 1). \tag{2.2.0.3}$$

We have following propositions.

Proposition 2.2.0.1. 1. $\rho \in P^+$ and therefore $\delta \in A$.

2. δ is skew-symmetric for W, i.e. $w\delta = \varepsilon(w)\delta$. Here we denote $\varepsilon(w) = (-1)^{\ell(w)}$ where $\ell(w)$ is length of $w \in W$ defined in Proposition 2.1.0.2.

3. Every skew-symmetric element $\xi \in A$ must be divisible by δ in A.

[2]Sometimes we may need to consider a bigger algebra such as $\frac{1}{2}\mathbb{Z}[P]$ for certain object to be considered valid but this is not a big issue.

Proof. (a) As s_i permutes the simple roots except α_i and $s_i(\alpha_i) = -\alpha_i$ so $s_i(\rho) = \rho - \alpha_i$. On the other hand, we have $s_i(\rho) = \rho - (\rho, \alpha_i^\vee)\alpha_i$ so $(\rho, \alpha_i^\vee) = 1$ for all i. This implies $\rho \in P^+$, which means $\delta \in A$.

(b) It suffices to prove for simple reflection s_i, and note from Proposition 2.1.0.2, s_i permutes $R^+ \setminus \{\alpha_i\}$ and $s_i(\alpha_i) = -\alpha_i$ so the only factor in (2.2.0.3) changes is e^ρ to $e^{\rho-\alpha_i}$ and $e^{-\alpha_i}$ to e^{α_i}. The net effect is that δ changes sign.

(c) First, note that δ is product of distinct irreducible elements $e^\alpha - 1$ in A, where α runs through R^+, times a unit $e^{-\rho}$. Hence, it suffices to show that ξ is divisible by each $1 - e^\alpha$. From property of ξ and of length ℓ, we know $s_\alpha(\xi) = -\xi$. Write $\xi = \sum_{\lambda \in P} n_\lambda e^\lambda$. Since $s_\alpha(\xi) = -\xi$ so $n_{s_\alpha(\lambda)} = -n_\lambda$. Noting that $s_\alpha(\lambda) = \lambda - k\alpha$ where $k = (\lambda, \alpha^\vee) \in \mathbb{Z}$, we find

$$\xi = \sum_{\lambda \in P, \lambda \mod \langle s_\alpha \rangle} n_\lambda (e^\lambda - e^{\lambda - k\alpha}).$$

Here the notation means we choose only one representative for each s_α orbit of P (orbit of P under s_λ only has two elements in $s_\lambda^2 = 1$ and note if $s_\alpha(\lambda) = \lambda$ then $n_\lambda = 0$). Since

$$e^\lambda - e^{\lambda - k\alpha} = (1 - e^\alpha)(-e^{\lambda-\alpha} - e^{\lambda - 2\alpha} - \cdots - e^{\lambda - k\alpha}).$$

So ξ is divisible by δ. $\square$

For each $\lambda \in P$, the sum

$$\sum_{w \in W} \varepsilon(w) e^{w(\lambda+\rho)}$$

is also skew-symmetric, hence by previous proposition, we find

$$\chi_\lambda = \delta^{-1} \sum_{w \in W} \varepsilon(w) e^{w(\lambda+\rho)} \tag{2.2.0.4}$$

is in A^W, called the *Weyl character* corresponding to λ. We define a partial order on V by $\mu < \lambda$ if $\lambda - \mu \in Q^+$ and $\lambda \neq \mu$.

We can also show that

$$\chi_\lambda = m_\lambda + \sum_{\mu < \lambda} K_{\lambda,\mu} m_\mu \tag{2.2.0.5}$$

where $K_{\lambda,\mu}$ are (positive) integers. This follows $\chi_\lambda, \lambda \in P^+$ form an $\mathbb{Z}$-basis of A^W.

Next, we define a scalar product on A. Let T be the torus $V/Q^\vee$, where $Q^\vee$ is the root lattice of $R^\vee$. The character group of T maybe identified with the weight lattice P, and we regard each $e^\lambda, \lambda \in P$ as character of T by the rule

$$e^\lambda(\dot{x}) = \exp 2\pi i(\lambda, x)$$

where $\dot{x} \in T$ is the image of $x \in V$, and exp is the exponential function. Then $f\bar{g}\Delta$ where $\Delta = \delta\delta^{-1}$ is a continuous function on the torus T, and we define a scalar product product $\langle , \rangle$ on A as

$$\langle f, g \rangle = |W|^{-1} \int_T f\bar{g}\Delta,$$

where the integration is taken with respect to normalized Haar measure.

Remark 2.2.0.2. This is a natural choice of inner product, reflecting from representation of compact Lie groups, where taking a (normalised) Haar integral over the whole compact Lie group G of a class function f on G boils down to taking Haar integral over its maximal torus T:

$$\int_G f(g)dg = \frac{1}{|W|}\int_T f(t)|\delta(t)|^2 dt,$$

here $\delta(t) = \sum_{w\in W} e^{w\rho} = e^{\rho}\prod_{\alpha\in R^+}(1-e^{-\alpha})$ and $W \cong N_G(T)/T$. This is the content of Weyl integration formula, which can be used to prove Weyl character formula for compact connected Lie groups.

2.3 Root system A_{n-1}

Here we take example of a root system $R = A_{n-1}$. Here we will derive the Schur polynomial and the monomial polynomial from root system A_{n-1}, subject to certain constraint:

(a) Consider $(n-1)$-dimensional vector space $V = \{v \in \mathbb{C}^n : v_1 + \cdots + v_n = 0\}$ with inner product $(a, b) = \sum_{i=1}^n a_i b_i$.

(b) Denote the root system $R = \{\varepsilon_i - \varepsilon_j\}_{i\neq j}$, where $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ where 1 in the ith place and the positive roots to be $R^+ = \{\varepsilon_i - \varepsilon_j\}_{i<j}$.

(c) The simple roots is then $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq n-1$. The corresponding coroots $\alpha_i^\vee = \alpha_i$.

(d) The Weyl group $W \cong S_n$ acts on V by permuting the components of $v \in V$. In particular, the simple reflections s_i are transpositions $(i, i+1)$.

(e) The root lattice $Q = \{\lambda \in \mathbb{Z}^n | \sum \lambda_i = 0\}$ as it is spanned by the simple roots.

(f) The weight lattice P. For $\beta \in P$, we need $\beta_i - \beta_{i+1} = (\beta, \alpha_i^\vee) \in \mathbb{Z}$.

(g) Dominant weights P^+: For $\beta \in P^+$, we need $\beta_i - \beta_{i+1} = (\beta, \alpha_i) \in \mathbb{Z}_{\geq 0}$. We also need $\sum \beta_i = 0$. We make a correspondence to partition: Here each partition $\mu = (\mu_1, \ldots, \mu_n)$ of length at most n determines a dominant weight λ by the rule $\lambda_i = \mu_i - n^{-1}|\mu|$ for $1 \leq i \leq n$ and two partitions μ, ν determine the same dominant weight if and only if $\mu_1 - \nu_1 = \cdots = \mu_n - \nu_n$. Thus, we convert condition of $\sum \lambda_i = 0$ to $\mu_1 - \nu_1 = \cdots = \mu_n - \nu_n$, and this later condition meant to say that we consider partition μ of length exactly n to be the same as partition when we remove all columns of size n from μ.

(h) For $\mathbb{C}[P]$, the group algebra of the weight lattice, spanned by the formal exponentials $e^\lambda, \lambda \in P$. Note that $\varphi_i = \varepsilon_i - \frac{1}{n}(\varepsilon_1 + \cdots + \varepsilon_n) \in P$. Put $x_i = e^{\varphi_i}$ for $1 \leq i \leq n$, then $x_1 \cdots x_n = 1$. Since W permutes the components, we find W permutes φ_i. In particular, an e^μ where $\mu = \sum_i c_i\varphi_i, c_i \in \mathbb{Q}$ is $x_1^{c_1}x_2^{c_2}\cdots x_n^{c_n}$. For positive root $\alpha = \varphi_i - \varphi_j = \varepsilon_i - \varepsilon_j$ with $i < j$ then $e^\alpha = x_i x_j^{-1}$.

(i) A root $\varepsilon_i - \varepsilon_j = \varphi_i - \varphi_j$ so we have $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha = \frac{1}{2}\sum_{i<j}(\varepsilon_i - \varepsilon_j)$. This follows

$$e^{\rho} = \left(\prod_{i<j} x_i x_j^{-1}\right)^{1/2} = \left((x_1 \cdots x_n)^{n-1} \prod_{i<j} x_i x_j^{-1}\right)^{1/2} = x_1^{n-1} \cdots x_{n-1}.$$

(j) Recall we assign for each partition $\mu = (\mu_1, \ldots, \mu_n)$ of length at most n a dominant weight λ by $\lambda_i = \mu_i - n^{-1}|\mu|$. Then $\lambda = \sum_i \mu_i \varphi_i$ so $e^{\lambda} = x_1^{\mu_1} \cdots x_n^{\mu_n}$. Thus, we obtain the monomial symmetric function m_λ in this case (subject to the condition $x_1 \cdots x_n = 1$). Similarly, one can obtain the Schur function from the Weyl characters.

Chapter 3

Connection with representation theory

3.1 Representation of Symmetric groups

Denote R_n the $\mathbb{Z}$-module generated by irreducible characters of S_n, and let $R = \bigoplus_{n\geq 0} R_n$. R has ring structure as follows:

For $f \in R_m, g \in R_n$ then $f \otimes g$ is a (virtual) character of $S_m \times S_n$. We view $S_m \times S_n$ as a subgroup of S_{m+n} by making S_m, S_n acting on $\{1\ldots, m\}$ and $\{m+1, \ldots, m+n\}$, respectively. This is a well-defined map up to conjugacy classes since conjugacy classes of S_n are classified by cycle-types. Hence, character $f \otimes g$ can be seen as a character of a subgroup $S_m \times S_n$ of S_{m+n}.

We define multiplication and comultiplication on R as follows: $m : R_m \otimes R_n \to R_{m+n}$ by

$$m(f \otimes g) = \mathrm{Ind}_{S_m \times S_n}^{S_{m+n}}(f \otimes g)$$

and $\Delta : R_n \to \bigoplus_{i+j=n} R_i \otimes R_j$ by

$$\Delta(f) = \sum_{i+j=n} \mathrm{Res}_{S_m \times S_n}^{S_{m+n}} f.$$

R also carries a scalar product inherited from inner product of class functions on S_n:

$$\langle f_1, f_2 \rangle_n = \frac{1}{|S_n|} \sum_{g \in S_n} f_1(g)\overline{f_2(g)}.$$

for $f, g \in S_n$. Therefore, we define scalar product on R as

$$\langle f, g \rangle = \sum_{n \geq 0} \langle f_n, g_n \rangle_n.$$

for $f, g \in R$ and $f = \sum f_n, g = \sum g_n$ with $f_n, g_n \in R_n$. One can show that R has a structure of a self-dual commutative, associative, graded, connected Hopf algebra.

Theorem 3.1.0.1. The *Frobenius map* ch is an isometric ring isomorphism of R onto $\Lambda_{\mathbb{C}}$, which can be defined by one of the following ways:

$$\begin{aligned} 1_{S_n} &\longmapsto h_n, \\ \mathrm{sgn}_{S_n} &\longmapsto e_n, \\ \chi^\lambda &\longmapsto s_\lambda, \\ \mathrm{Ind}_{S_\lambda}^{S_n} 1_{S_\lambda} &\longmapsto h_\lambda, \\ \mathrm{Ind}_{S_\lambda}^{S_n} \mathrm{sgn}_{S_\lambda} &\longmapsto e_\lambda, \\ 1_\lambda &\longmapsto \frac{p_\lambda}{z_\lambda}, \\ f \in R_n &\longmapsto \sum_{|\rho|=n} z_\rho^{-1} f(\rho) p_\rho. \end{aligned}$$

Here $1_\lambda, \mathrm{sgn}_{S_n}$ are the trivial, sign characters restricted to $S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots$; χ^λ is the irreducible characters of S_n correpending to partition λ of n; $1_\lambda \in R_n$ is the characteristic function for the S_n-conjugacy class of cycle type λ; $z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!$ where $m_i(\lambda)$ is the multiplicity of i in λ.

From this, one follows that the transition matrix $M(p,s)$ from the power sums in $\Lambda_{\mathbb{C}}$ to the Schur functions is the character table of S_n's. In other words, the irreducible characters χ^λ evaluated at partition/ cycle type/conjugacy class ρ is $\chi^\lambda_\rho = M(p,s)_{\rho\lambda}$, i.e.

$$p_\rho = \sum_\lambda \chi^\lambda_\rho s_\lambda.$$

3.1.1 Construction of simple $\mathbb{C}[S_n]$-modules

These computations of characters suggest a simple way to construct irreducible representation of S_n. We have two following identities in Λ, obtained by looking at the transition matrices between bases of Λ:

$$\begin{aligned} h_\lambda &= s_\lambda + \sum_{\mu > \lambda} K_{\mu\lambda} s_\mu, \\ e_{\lambda'} &= s_\lambda + \sum_{\mu < \lambda} K_{\mu'\lambda'} s_\mu. \end{aligned}$$

Hence, in R, we have

$$\begin{aligned} \mathrm{Ind}_{S_\lambda}^{S_n} 1_{S_\lambda} &= \chi^\lambda + \sum_{\mu > \lambda} K_{\mu\lambda} \chi^\mu, \\ \mathrm{Ind}_{S_{\lambda'}}^{S_n} \mathrm{sgn}_{S_{\lambda'}} &= \chi^\lambda + \sum_{\mu < \lambda} K_{\mu'\lambda'} \chi^\mu. \end{aligned}$$

This follows that there is a common submodule of $\mathrm{Ind}_{S_\lambda}^{S_n} 1_\lambda$ and $\mathrm{Ind}_{S_{\lambda'}}^{S_n} \mathrm{sgn}_{S_\lambda}$, which is a simple $\mathbb{C}[S_n]$-module with character χ^λ. Hence, it suffices to find a common element in both of these $\mathbb{C}[S_n]$-modules and then we can generate to obtain a simple $\mathbb{C}[S_n]$-module:

Let λ be a partition and let T be any numbering of λ with $1, 2, \ldots, n$. Let R (resp. C) denote the subgroup of S_n that stabilizes each row (resp. column) of T, so that $R \cong S_\lambda$ and $C \cong S_{\lambda'}$. Let $A = \mathbb{C}[S_n]$ be the group algebra of S_n and let

$$a = \sum_{u \in C} \varepsilon(u) u, s = \sum_{v \in R} v.$$

One can show As is the induced module $\mathrm{Ind}_R^{S_n}(1_{S_\lambda})$ and likewise $Aa \cong \mathrm{Ind}_{S_{\lambda'}}^{S_n} \mathrm{sgn}_{S_{\lambda'}}$. Let $e = as \in A$ then note $M_\lambda = Ae$ is a submodule of As and it is isomorphic to a quotient of Aa under homomorphism $x \mapsto xs$ $(x \in A)$. Hence, M_λ is the submodule appearing in both $\mathrm{Ind}_R^{S_n}(1_{S_\lambda})$ and $\mathrm{Ind}_{S_{\lambda'}}^{S_n} \mathrm{sgn}_{S_{\lambda'}}$. Therefore, from the above observation, we find M_λ is the irreducible S_n-module with character χ^λ.

Example 3.1.1.1. Take $\lambda = (n)$ then $R \cong S_n$ and C is just identity element. Hence, $as = \sum_{v \in S_n} v$ and hence $M_{(n)} = \mathbb{C}_{\mathrm{sym}}$, the trivial representation.

Take $\lambda = (1^n)$ then $C \cong S_n$ and R is just identity element. Hence, $as = \sum_{w \in S_n} \varepsilon(w) w$. We find $M_{(1^n)} = \mathbb{C}_{\mathrm{alt}}$, the alternating representation, i.e. $\mathbb{C}$ with $\sigma x = \varepsilon(\sigma) x$ for $x \in \mathbb{C}$. ⌟

3.1.2 Murnaghan–Nakayama rule

First, one can show that [10, Chapter 1, §3, exercise 11]

$$s_\mu p_r = \sum_\lambda (-1)^{\mathrm{ht}(\lambda - \mu)} s_\lambda \tag{3.1.2.1}$$

summed over all partitions $\lambda \supset \mu$ such that $\lambda - \mu$ is a border strip of length r. Recall that the height ht of a border strip is one less than number of rows it occupies.

From this, we obtain a combinatorial rule to compute χ^λ_ρ, which is the coefficient of s_λ in p_ρ:

$$\chi^\lambda_\rho = \sum_S (-1)^{\mathrm{ht}(S)}$$

summed over all sequence of partitions $S = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(m)})$ such that $m = l(\rho)$, $0 = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(m)} = \lambda$, and such that each $\lambda^{(i)} - \lambda^{(i-1)}$ is a border strip of length ρ_i, and $\mathrm{ht}(S) = \sum_i \mathrm{ht}(\lambda^{(i)} - \lambda^{(i-1)})$.

3.1.3 Branching rules

In general, for subgroup H of G, then *branching rule* from G to H is a description of how irreducible characters of G decomposes as irreducible characters of H. Here we will discuss branching rule from S_n to S_{n-1}. Recall that the Frobenius character F satisfies the following multiplicative property:

$$\mathrm{ch}\left(\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} f \otimes g\right) = \mathrm{ch}(f)\mathrm{ch}(g), \tag{3.1.3.1}$$

where f, g are characters of S_m, S_n, respectively. This follows for λ a partition of n, we have
$$\mathrm{ch}(\mathrm{Ind}_{S_n}^{S_{n+1}}\chi^\lambda) = \mathrm{ch}(\chi^\lambda)\mathrm{ch}(1_{(1)}) = s_\lambda p_1.$$
Hence, this is a special case of Murnaghan–Nakayama rule, which gives us the decomposition of irreducible S_n-module M_λ when restricting to S_{n-1}:
$$M_\lambda = \bigoplus_\mu M_\mu$$
summed over all $\mu \subset \lambda$ and $|\mu| = |\lambda| - 1$.

3.2 Duality between GL$(n, \mathbb{C})$ and S_k

Let V be complex vecor space and $V^{\otimes k}$ be k-fold tensor over $\mathbb{C}$. Consider right $\mathbb{C}[S_k]$-module structure of $V^{\otimes k}$, where $\sigma \in S_k$ acts by permuting the factors:
$$(v_1 \otimes \cdots v_k)\sigma = v_{\sigma(1)} \otimes \cdots v_{\sigma(k)}.$$
It also has GL(V)-module structure $a \in \mathrm{GL}(V)$ acts diagonally:
$$a(v_1 \otimes \cdots v_k) = av_1 \otimes \cdots av_k.$$
The two actions commutes with each other, making $V^\otimes$ into an $(\mathrm{GL}(V), \mathbb{C}[S_k])$-bimodule. Hence, if V_λ is an $\mathbb{C}[S_k]$-module then
$$S_\lambda(V) := V^{\otimes k} \otimes_{\mathbb{C}[S_k]} V_\lambda$$
is a left GL(V)-module [1].

One can show that (via Double Centralizer Theorem), for V_λ a simple $\mathbb{C}[S_n]$-module, then $S_\lambda(V)$ is a simple $GL(V)$-module, whose characters is the Schur polynomial $g \mapsto s_\lambda(t_1, \ldots, t_n)$ where $(t_1, \ldots, t_n)$ are the eigenvalues of g.

For example, when $V_{(n)} = \mathbb{C}_{\mathrm{sym}}$ then $S_{(n)}(V) \cong \mathrm{Sym}^{\otimes k} V$ and when $V_{(1^n)} = \mathbb{C}_{\mathrm{alt}}$ then $S_{(1^n)}(V) \cong \bigwedge^{\otimes k} V$ from example 3.1.1.1.

In fact, if we define an *algebraic* (or rational or polynomial) representation V of $GL(V)$ if its character are polynomial functions over the matrix coefficients and $(\det)^{-1}$. Then $S_\lambda(V)$ where $l(\lambda) \leq n$ are all the irreducible algebraic representations of $GL(V)$ and algebraic representations are semisimple.

Furthermore, being an algebraic representation is the same as being an analytic representation of $GL(V)$. This follows that $S_\lambda(V)$ are also irreducible representations of $U(V)$. It seems there are more connection to this, but we have not gone through the full details to understand what is going on.

[1] When R is not necessary commutative, given R-modules M, N then $M \otimes_R N$ does not generally has R-module structure. If A is another ring, W is an (A, R)-*bimodule* if it has left A-module and a right R-module structure, such that these structures commute in the sense that $m \in M, a \in A, r \in R$ then $a(mr) = (am)r$. If M is (A, R)-bimodule, then $M \otimes_R N$ is left A-module with $a(m \otimes n) = am \otimes n$ for $a \in A$

Bibliography

[1] Butler, L., 1995. *Subgroup Lattices And Symmetric Functions*. Amer Mathematical Society.

[2] Garsia, A. *Orthogonality of Milne's polynomials and raising operators*, Discrete Mathematics, Volume 99, Issues 1–3, 1992, Pages 247-264, ISSN 0012-365X, https://doi.org/10.1016/0012-365X(92)90375-P

[3] Grinberg, D., Reiner, V. *Hopf algebras in Combinatorics* http://www.cip.ifi.lmu.de/~grinberg/algebra/HopfComb.pdf

[4] Haglund, J., 2008. *The q,t-Catalan Numbers And The Space Of Diagonal Harmonics*. Providence: American Mathematical Society.

[5] Jing, N. *Vertex operators and Hall-Littlewood symmetric functions*, Advances in Mathematics, Volume 87, Issue 2, 1991, Pages 226-248, ISSN 0001-8708, https://doi.org/10.1016/0001-8708(91)90072-F.

[6] Knutson, D., 1973. *Lambda-Rings And The Representation Theory Of The Symmetric Group*. Berlin 1973.

[7] Zelevinsky, A., 1981. *Representations Of Finite Classical Groups*. Berlin: Springer.

[8] Lascoux, A., 2003. *Symmetric Functions And Combinatorial Operators On Polynomials*. Providence, R.I.: American Mathematical Society.

[9] Loehr, N.A., Remmel, J.B. *A computational and combinatorial exposé of plethystic calculus*. J Algebr Comb 33, 163–198 (2011). https://doi.org/10.1007/s10801-010-0238-4

[10] Macdonald I. G. (1995), *Symmetric functions and Hall polynomials*, 2nd edition, Oxford Univ. Press, New York.

[11] Warnaar, O. Lecture notes on Symmetric Functions. Virginia Integrable Probability Summer School 2019.