1.3.3. Localization of rings and modules

- A ring with 1. A multiplicative subset S of A is subset closed under multiplication containing 1. We define ring $S^{-1}A$ with ring hom $A \to S^{-1}A$ satisfy following universal property:

  Any ring hom $f: A \to B$ (in other words, B is A-algebra) where every element of S is sent to an invertible element must factor uniquely through $A \to S^{-1}A$, i.e. if $f: A \to B$ so $f(s)$ invertible for all $s \in A$ then $\exists g: S^{-1}A \to B$ s.t. $f = g\phi$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi"'] & B \\ S^{-1}A \arrow[ur, dashed, "\exists! g"'] & \end{tikzcd}$$

- Different way to say this: $S^{-1}A$ is initial among A-algebra B where every element of S is sent to an invertible element of B.

- Localization of modules: Suppose M is A-module. We define A-module map $\phi: M \to S^{-1}M$ as being initial among A-modules $M \to N$ s.t. elements of S are invertible in N ($s \times \cdot : N \to N$ is isomorphism for all $s \in S$). More precisely, any such map $\alpha: M \to N$ factor uniquely through ϕ:

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[dr, "\alpha"'] & S^{-1}M \arrow[d, dashed, "\exists!"] \\ & N \end{tikzcd}$$

  $S^{-1}M$ can be viewed as $S^{-1}A$-module (an $S^{-1}A$-module is just A-module but action of s gives an automorphism).

§1.6. An introduction to abelian categories

$\boxed{1.6F}$ Suppose A ring, $S \subset A$ multiplicative subset of A, M is A-module

(a) Localization of A-modules $\mathrm{Mod}_A \to \mathrm{Mod}_{S^{-1}A}$ is exact covariant functor.

(b) $(\cdot) \otimes_A M$ is right-exact covariant functor $\mathrm{Mod}_A \to \mathrm{Mod}_A$.

(c) $\mathrm{Hom}(M, \cdot)$ is left-exact covariant functor $\mathrm{Mod}_A \to \mathrm{Mod}_A$. If $\mathscr{C}$ is abelian category, and $C \in \mathscr{C}$, show $\mathrm{Hom}(C, \cdot)$ is left-exact covariant functor $\mathscr{C} \to Ab$.

(d) Show $\mathrm{Hom}(\cdot, M)$ is left-exact contravariant functor $\mathrm{Mod}_A \to \mathrm{Mod}_A$. If $\mathscr{C}$ is any abelian category, and $C \in \mathscr{C}$, show $\mathrm{Hom}(\cdot, C)$ is left-exact contravariant functor $\mathscr{C} \to Ab$.

Chp1: Category theory

§1.3. Universal properties

1.35

$$\begin{tikzcd}
X_1\times_Y X_2 \arrow[r, "\phi"] \arrow[d, "\beta"'] & X_1\times_Z X_2 \arrow[d, "\alpha"] \\
Y \arrow[r, "i_Y"'] & Y\times_Z Y
\end{tikzcd}$$

First, we try to find relations between ϕ, α, β, i_Y by recalling universal prop of $X_1\times_Z X_2$, $X_1\times_Y X_2$, $Y\times_Z Y$

$$\begin{tikzcd}
X_1\times_Y X_2 \arrow[rrd, bend left, "x_{y_1}"] \arrow[rdd, bend right, "x_{y_2}"'] \arrow[rd, dashed, "\phi"] & & & \\
& X_1\times_Z X_2 \arrow[r, "x_{z_1}"] \arrow[d, "x_{z_2}"'] & X_1 \arrow[d, "x_1"] & \\
& X_2 \arrow[r, "x_2"'] & Y \arrow[rd, "y"] & \\
& & & Z
\end{tikzcd}$$

According to universal prop, $\exists\, x_{y_1} : X_1\times_Y X_2 \to X_1$

$x_{y_2} : X_1\times_Y X_2 \to X_2$

and $x_{z_1} : X_1\times_Z X_2 \to X_1$

$x_{z_2} : X_1\times_Z X_2 \to X_2$ so $x_1 x_{y_1} = x_2 x_{y_2} =: \beta$ (1)

$y x_1 x_{z_1} = y x_2 x_{z_2}$

then ϕ will satisfies : $x_{z_2}\phi = x_{y_2}$; $x_{z_1}\phi = x_{y_1}$. (2)

$$\begin{tikzcd}
X_1\times_Z X_2 \arrow[r, "x_{z_1}"] \arrow[rd, dashed, "\alpha"] \arrow[dd, "x_{z_2}"'] & X_1 \arrow[rd, "x_1"] & \\
& Y\times_Z Y \arrow[r, "y_1"] \arrow[d, "y_2"] & Y \arrow[d, "y"] \\
X_2 \arrow[r, "x_2"'] & Y \arrow[r, "y"'] & Z
\end{tikzcd}$$

By universal prop of $Y\times_Z Y$, $\exists\, y_1, y_2 : Y\times_Z Y \to Y$

so $y y_1 = y y_2$. (3)

α will satisfies $y_2\alpha = x_2 x_{z_2}$, $y_1\alpha = x_1 x_{z_1}$ (4)

$$\begin{tikzcd}
Y \arrow[rrd, bend left, "1_Y"] \arrow[rdd, bend right, "1_Y"'] \arrow[rd, dashed, "i_Y"] & & \\
& Y\times_Z Y \arrow[r, "y_1"] \arrow[d, "y_2"] & Y \arrow[d, "y"] \\
& Y \arrow[r, "y"'] & Z
\end{tikzcd}$$

By universal properties of $Y\times_Z Y$, then $i_Y : Y \to Y\times_Z Y$ satisfies $y_1 i_Y = y_2 i_Y = 1_Y$ identity on Y. (5)

⊛ Back to our problem, we want to show first that the following commutes

$$\begin{tikzcd}
X_1\times_Z X_2 \arrow[r, "\phi"] \arrow[d, "\beta"'] & X_1\times_Y X_2 \arrow[d, "\alpha"] \\
Y \arrow[r, "i_Y"'] & Y\times_Z Y
\end{tikzcd}$$

it suffices to show $i_Y\beta$ and $\alpha\beta$ are maps to $Y_1\times_Z Y$ that satisfies universal prop of $Y_1\times_Z Y$

$$\begin{tikzcd}
X_1\times_Y X_2 \arrow[rrd, bend left, "x_1 x_{y_1}"] \arrow[rdd, bend right, "x_2 x_{y_2}"'] \arrow[rd, dashed, "\alpha\phi"] \arrow[rd, dashed, "i_Y\beta"'] & A & \\
B & Y\times_Z Y \arrow[r, "y_1"] \arrow[d, "y_2"] & Y \arrow[d, "y"] \\
& Y \arrow[r, "y"'] & Z
\end{tikzcd}$$

Check triangle A commutes with $\alpha\phi$; i.e. need $y_1\alpha\phi = x_1 x_{y_1}$

We have $y_1\alpha\phi = x_1 x_{z_1}\phi$ (from (4))

$= x_1 x_{y_1}$ (from (2))

Check triangle A commutes (with $i_Y\beta$):

We have $y_1 i_Y\beta = y_1 i_Y x_1 x_{y_1}$ (from (1))

$= x_1 x_{y_1}$ (from (5))

Due to uniqueness of $X_1\times_Y X_2 \to Y_1\times_Z Y$ so we find $i_Y\beta = \alpha\phi$, as desired.

⊛ We prove uniqueness properties of magic square:

$$\begin{tikzcd}
W \arrow[rrd, bend left, "w_x"] \arrow[rdd, bend right, "w_y"'] \arrow[rd, dashed, "\varphi"] & & \\
& X_1\times_Y X_2 \arrow[r, "\phi"] \arrow[d, "\beta"] & X_1\times_Z X_2 \arrow[d, "\alpha"] \\
& Y \arrow[r, "i_Y"'] & Y\times_Z Y
\end{tikzcd}$$

Suppose $W \in \mathcal{C}$ and w_x, w_y as in diagram

so $\alpha w_x = i_Y w_y$.

$\Rightarrow y_1\alpha w_x = y_2\alpha w_x \,(= y_1 i_Y w_y = y_2 i_Y w_y) \overset{(5')}{=} w_y$ (see (5))

$\Rightarrow x_1 x_{z_1} w_x = x_2 x_{z_2} w_x$ (see (4))

$$\begin{tikzcd}
W \arrow[rrd, bend left, "x_{z_1}w_x"] \arrow[rdd, bend right, "x_{z_2}w_x"'] \arrow[rd, dashed, "\varphi"] & & \\
& X_1\times_Y X_2 \arrow[r, "x_{y_1}"] \arrow[d, "x_{y_2}"'] & X_1 \arrow[d, "x_1"] \\
& X_2 \arrow[r, "x_2"'] & Y
\end{tikzcd}$$

$\Rightarrow$ the diagram on the left

$$\begin{tikzcd}
W \arrow[r, dash] \arrow[d, dash] & X_1 \arrow[d, dash] \\
X_2 \arrow[r, dash] & Y
\end{tikzcd}$$

commutes

$\Rightarrow$ by u.p of $X_1\times_Y X_2$, $\exists\,\varphi : W \to X_1\times_Y X_2$ s.t.

$x_{y_1}\varphi = x_{z_1}w_x$, $x_{y_2}\varphi = x_{z_2}w_x$. (6)

⊛ We show this φ is our desired map, i.e. show $\phi\varphi = w_x$, $\beta\varphi = w_y$:

- We have $x_{z_1}\phi\,\varphi \overset{(2)}{=} x_{y_1}\varphi \overset{(6)}{=} x_{z_1}w_x$. and similar $x_{z_2}\phi\varphi = x_{z_2}w_x$.

so by universal prop of $X_1\times_Z X_2$ so $\phi\varphi = w_x$.

- We have $\beta\varphi = x_1 x_{y_1}\varphi \overset{(6)}{=} x_1 x_{z_1} w_x \overset{(4)}{=} y_1\alpha w_x \overset{(5')}{=} w_y$.

We are done. ▥

Chapter 2: Sheaves

§2.2. Definition sheaf and presheaf

$\boxed{2.2\text{I}}$ Suppose $\pi: X \to Y$ continuous map, $\mathcal{F}$ sheaf of sets (or rings or A-modules) on X. If $\pi(p) = q$, describe the natural morphism of stalks $(\pi_* \mathcal{F})_q \to \mathcal{F}_p$.

– Description using representatives of stalks:

$f \in (\pi_* \mathcal{F})(U) = \mathcal{F}(\pi^{-1}(U))$ with $q \in U$. Then as $\pi(p) = q$ so $p \in \pi^{-1}(U)$
so we can restrict $\mathcal{F}(\pi^{-1}(U))$ to open set containing p to get element of $\mathcal{F}_p$.

– Description using universal property.

We have $(\pi_* \mathcal{F})_q = \varinjlim_{q \in U} \mathcal{F}(\pi^{-1}(U))$ and $\mathcal{F}_p = \varinjlim_{p \in V} \mathcal{F}(V)$.

Note $p \in \pi^{-1}(U)$ so this induces $\mathcal{F}(\pi^{-1}(U)) \to \mathcal{F}_p$. $\forall$ open $U \ni q$.

By universal property of $(\pi_* \mathcal{F})_q$, exists map $(\pi_* \mathcal{F})_q \to \mathcal{F}_p$ s.t.

$$\begin{tikzcd}
\mathcal{F}(\pi^{-1}(U)) \arrow[d] \arrow[dr] & \\
(\pi_* \mathcal{F})_q \arrow[r] & \mathcal{F}_p
\end{tikzcd}$$

commutes for all open $U \ni q$.

2.3J $X = \mathbb{C}$ with classical topology, $\underline{\mathbb{Z}}$ constant sheaf on X associated to $\mathbb{Z}$.

$\mathcal{O}_X$ sheaf of holomorphic functions

$\mathcal{F}$ presheaf of functions admitting holomorphic logarithm.

Describe exact sequence of presheaves on X:

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \to \mathcal{F} \to 0$$

$$f \longmapsto \exp(2\pi i f).$$

$$0 \to \underline{\mathbb{Z}}(U) \longrightarrow \mathcal{O}_X(U) \longrightarrow \mathcal{F}(U) \longrightarrow 0$$

$$\left\{\begin{matrix}\text{continuous}\\ f: U \to \mathbb{Z}\end{matrix}\right\} \mapsto \{f: U \to \mathbb{Z} \subset \mathbb{C}\}$$

$$(f: U \to \mathbb{C}) \mapsto \exp(2\pi i f): U \to \mathbb{C}^\times$$

$\mathcal{F}$ is not a sheaf. Because gluability fails. Consider open set $\mathbb{C} \setminus \{0\}$, $z \mapsto z$ admits no logarithm but it has locally logarithm ($z = e^{\log z}$ as long as

$\boxed{2.4E}$ (Isomorphisms are determined by stalks)

Show that morphism of sheaves of sets is an isomorphism iff it induces isomorphisms for all stalks.

"$\Rightarrow$": Given isomorphism $\phi(U): \mathcal{F}(U) \to \mathcal{G}(U)$ for all open $U \subset X$. Show $\phi_p: \mathcal{F}_p \to \mathcal{G}_p$ is an isomorphism for all $p \in U$.

- Injectivity of ϕ_p: Suppose $s_p = (s, U_p) \in \mathcal{F}_p$, $t_p = (t, U_p) \in \mathcal{F}_p$ so $\phi_p(s_p) = \phi_p(t_p)$ or $\phi(s)_p = \phi(t)_p$ then $\phi(s)|_{W_p} = \phi(t)|_{W_p}$ for some open $W_p \ni p$ $\Rightarrow$ $\phi(s|_{W_p}) = \phi(t|_{W_p})$ so by injectivity of $\phi(W_p)$, we find $s|_{W_p} = t|_{W_p}$ $\Rightarrow$ $s_p = t_p$ as $p \in W_p$.
- Surjectivity: For $t_p = (t, U_p) \in \mathcal{G}_p$, as $\phi(U_p)$ isomorphism, exists $s \in \mathcal{F}(U_p)$ so $\phi(s) = t$. $\Rightarrow t_p = \phi_p(s_p)$.

Thus, $\phi_p: \mathcal{F}_p \to \mathcal{G}_p$ is an isomorphism for all $p \in U$

"$\Leftarrow$": Given $\phi_p: \mathcal{F}_p \to \mathcal{G}_p$ isomorphism for all $p \in U$. Show $\phi(U): \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism of sets.

- Injectivity of $\phi(U)$: This follows from the injectivity $\mathcal{F}(U) \hookrightarrow \prod_{p \in U} \mathcal{F}_p$ for any sheaf $\mathcal{F}$, i.e. we have commutative diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "(*)"] & \mathcal{G}(U) \arrow[d, hook] \\
\prod_{p \in U} \mathcal{F}_p \arrow[r, "\sim"] & \prod_{p \in U} \mathcal{G}_p
\end{tikzcd}$$

Note $\prod_{p \in U} \mathcal{F}_p$ and $\prod_{p \in U} \mathcal{G}_p$ isomorphic as from assumption. This follows $\mathcal{F}(U) \to \mathcal{G}(U)$ injective.

- Surjectivity of $\phi(U)$: Pick $t \in \mathcal{G}(U)$. By surjectivity of ϕ_p, there exists $s_p = (s^{W_p}, W_p) \in \mathcal{F}_p$ so $\phi_p(s_p) = \phi(s^{W_p})_p = t_p$.
  - We show that $\prod_{p \in U} s_p$ is a compatible germ, i.e. exists a section s of $\mathcal{F}(U)$ whose image is this. Then $(*)$ will follow $\phi(s) = t$. (as $\mathcal{G}(U) \hookrightarrow \prod_{p \in U} \mathcal{G}_p$ injective).
  - Indeed, it suffices to show $(s^{W_p})_q = s_q$ for all $q \in W_p$.

    As $\phi(s^{W_p})_p = t_p$ so $\phi(s^{W_p}) = t|_{W_p}$ in $\mathcal{G}(W_p)$.

    $\overset{q \in W_p}{\Rightarrow}$ $\phi(s^{W_p})_q = t_q \in \mathcal{G}_p$. We also have $\phi_q(s_q) = t_q$

    $\Rightarrow \phi_q(s^{W_p}{}_q) = \phi_q(s_q) \Rightarrow (s^{W_p})_q = s_q$ (by injectivity of ϕ_q)

    so surjective of ϕ_p is not enough to show $\phi(U)$ surjective

    $\Rightarrow \prod_{p \in U} s_p$ is a compatible germ, as desired.

$\boxed{2.4F}$ (a) Injectivity $\mathcal{F}(U) \hookrightarrow \prod_{p \in U} \mathcal{F}_p$ only uses the identity axiom in definition of sheaf $\mathcal{F}$. So here is example where it is false for general presheaf:

$X = \{a, b\}$ with discrete topology

$\mathcal{F}(a) = \{*\}$, $\mathcal{F}(b) = \{*\}$, $\mathcal{F}(\emptyset) = \{*\}$, $\mathcal{F}(\{a,b\}) = \{x, y\}$

with the restriction map all mapping to $*$.

- Then $x \mapsto (x_a, x_b) = (*, *) \leftarrow y$ but $x \neq y$ so $\mathcal{F}(\{a,b\}) \to \mathcal{F}_a \times \mathcal{F}_b$ not injective

(b) Choose $\mathcal{G}$ to be as in (a) and $\phi_1, \phi_2 : \mathcal{F}^{\#} \to \mathcal{G}$ (any so $\phi_1 \neq \phi_2$)

We have $\phi_{B1} = \phi_{B2}$ from def of $\mathcal{G}$.

(c) not hard, follow the two above example

$\boxed{2.4G}$ Show sheafification unique up to unique isomorphism

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "sh1"] \arrow[dr, "sh2"'] & \mathcal{F}^{sh1} \arrow[d, dashed, shift left, "\exists! f"] \\
& \mathcal{F}^{sh2} \arrow[u, dashed, shift left, "\exists! g"]
\end{tikzcd}$$

$$\left.\begin{array}{l} f \circ sh1 = sh2 \\ g \circ sh2 = sh1 \end{array}\right\} \Rightarrow (fg) \circ sh2 = sh2$$

$\Rightarrow fg = id_{\mathcal{F}^{sh2}}$

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "sh2"] \arrow[dr, "sh2"'] & \mathcal{F}^{sh2} \arrow[d, "id"', "fg"] \\
& \mathcal{F}^{sh2}
\end{tikzcd}$$

by universal prop of $\mathcal{F} \xrightarrow{sh2} \mathcal{F}^{sh2}$.

Similarly $gf = id_{\mathcal{F}^{sh1}}$.

$\boxed{2.4H}$ Sheafification is a functor. preshf $\to$ Shf.

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\phi"] \arrow[d, "sh"'] & \mathcal{G} \arrow[d, "sh_{\mathcal{G}}"] \\
\mathcal{F}^{sh} \arrow[r, "\phi^{sh}"] & \mathcal{G}^{sh}
\end{tikzcd}$$

Since $sh_{\mathcal{G}} \circ \phi : \mathcal{F} \to \mathcal{G}^{sh}$ morphism so $\exists! \ \phi^{sh} : \mathcal{F}^{sh} \to \mathcal{G}^{sh}$ so the left diagram commutes.

and

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\phi"] \arrow[d, "sh"'] & \mathcal{G} \arrow[r, "\alpha"] \arrow[d, "sh"] & \mathcal{H} \arrow[d, "sh"] \\
\mathcal{F}^{sh} \arrow[r, "\phi^{sh}"'] & \mathcal{G}^{sh} \arrow[r, "\alpha^{sh}"'] & \mathcal{H}^{sh}
\end{tikzcd}$$

Two small $\square$ commutes so large square commutes, i.e. $\alpha^{sh}\phi^{sh}sh_{\mathcal{F}} = sh_{\mathcal{H}}\alpha\phi$

By def, $(\alpha\phi)^{sh}$ also makes the large diagram commutes and this morphism is unique in that condition.

$\Rightarrow \alpha^{sh}\phi^{sh} = (\alpha\phi)^{sh}$

Thus, sheafification is a functor.

Also, note this functor sends $sh: \mathcal{F} \to \mathcal{F}^{sh}$ to the identity $\mathcal{F}^{S} \to \mathcal{F}^{S}$.

h h

$\boxed{2.4I}$ Show $\mathcal{F}^{sh}$ (constructed in 2.4.7 for presheaf of sets with additional structure). forms a sheaf.

- The restriction map: $res_{V \supset U} : (f_p \in \mathcal{F}_p)_{p \in V} \mapsto (f_p \in \mathcal{F}_p)_{p \in U}$

⊛ Show it's a presheaf: easily follows from the restriction map

⊛ Identity axiom: $s,t \in \mathcal{F}^{sh}(U)$ where $s = (s_p)_{p \in U}, t = (t_p)_{p \in U}$ and $s|_{U_i} = t|_{U_i} \Rightarrow s_p = t_p \ \forall p \in U_i$. This holds for all i, so $s = t$ as $\{U_i\}$ open cover of U

⊛ Gluability axiom: $s_i = (s_{i,p})_{p \in U_i} \in \mathcal{F}^{sh}(U_i)$ s.t. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \Rightarrow s_{i,p} = s_{j,p} \ \forall p \in U_i \cap U_j$.

We show $s = (s_p)_{p \in U}$ where $s_p = s_{i,p}$ if $p \in U_i$ satisfies $s|_{U_i} = s_i$. This is true obviously from the restriction map.

- But we need also to show $s \in \mathcal{F}^{sh}(U)$. But this also follows from the local def of elements of $\mathcal{F}^{sh}(U)$.

$\boxed{2.4J}$ $sh: \mathcal{F} \to \mathcal{F}^{sh}$ is map of presheaf, i.e. for $f \in \mathcal{F}(U)$

$Sh(U) := f \in \mathcal{F}(U) \mapsto (f_p)_{p \in U}$

then $sh(f|_V) = sh(f)|_V = (f_p)_{p \in V}$

$\boxed{2.4K}$ sh defined in 2.4J satisfies the universal properties

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "sh"] \arrow[dr, "\phi"'] & \mathcal{F}^{sh} \arrow[d, dashed, "\exists! \psi"] \\
& \mathcal{G}
\end{tikzcd}$$

Consider sheaf $\mathcal{G}$ and presheaf hom $\mathcal{F} \to \mathcal{G}$.

⊛ Define $\psi(U)$: $(s_p \in \mathcal{F}_p)_{p \in U} = s \in \mathcal{F}^{sh}(U)$.

- From definition, we know for each p, exists (f^p, U_p) with $f^p \in \mathcal{F}(U_p)$, $p \in U_p$ so $(f^p)_q = s_q \ \forall q \in U_p$.

$\Rightarrow f^p|_{U_p \cap U_q} = f^q|_{U_p \cap U_q}$ (as for any $t \in U_p \cap U_q$, restriction to same neighborhood of t makes them the same)

Consider $\phi(f^p) \in \mathcal{G}(U_p)$ for all $p \in U$. We have

$$\phi(f^p)|_{U_p \cap U_q} = \phi(f^p|_{U_p \cap U_q}) = \phi(f^q|_{U_p \cap U_q}) = \phi(f^q)|_{U_p \cap U_q}$$

so by gluability of $\mathcal{G}$, exists $f \in \mathcal{G}(U)$ so $f|_{U_p} = f^p$.

In other words $(\phi(f^p))_{p \in U}$ is compatible germs in $\mathcal{G}(U)$ whose inverse image under injection $\mathcal{G}(U) \to \prod_{p \in U} \mathcal{G}_p$ is f.

- We then define $\psi(s) = f$. From the construction, we find the commutative diagram commutes.
- It is a presheaf morphism. Not hard to see from the construction

2.4L Sheaffication is the left adjoint to the forgetful functor from sheaves on X to presheaves on X.

$$\mathrm{Mor}_{sh}(\mathcal{F}^{sh}, \mathcal{G}) \simeq \mathrm{Mor}_{presh}(\mathcal{F}, \mathcal{G})$$

- Denote $sh: \mathcal{F} \to \mathcal{F}^{sh}$ then $g \in \mathrm{Mor}(\mathcal{F}^{sh}, \mathcal{G}) \mapsto g \circ sh$

- For the inverse consider $f \in \mathrm{Mor}_{presh}(\mathcal{F}, \mathcal{G})$ then $f \mapsto f^{sh}$.

- Show the two are inverse of each other. We recall def of f^{sh} in 2.4H.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "f"] \arrow[d, "sh"'] & \mathcal{G} \arrow[d, "id"] \\ \mathcal{F}^{sh} \arrow[r, "f^{sh}"'] & \mathcal{G} \end{tikzcd}$$

which is the unique sheaf morphism making the left diagram commutes.

$\Rightarrow \quad f^{sh} \circ sh = f$, as desired.

2.4M $sh: \mathcal{F} \to \mathcal{F}^{sh}$ induces isomorphism of stalks.

Show $sh_p: \mathcal{F}_p \to \mathcal{F}_p^{sh}$ is an isomorphism.

Consider the inverse $\mathcal{F}_p^{sh} \to \mathcal{F}_p$ defined as for $p \in U$, $(s_q \in \mathcal{F}_q)_{q \in U} \mapsto s_p$

⊛ Sheafification of constant presheaf is corresponding constant sheaf:

Consider constant presheaf $\underline{S}_{pre}$ associated to S. Then $\underline{S}_{pre}(U) = S$ $\forall$ open $U \subseteq X$.

Its sheafification is $\underline{S}_{pre}^{sh}(U) = \{(s_p)_{p \in U}$ compatible germs$\}$.

$(s_p)_{p \in U}$ compatible mean for any $p \in U$, exists open $V \subset U$, $p \in V$ and $f \in \underline{S}_{pre}(V) = S$ s.t. $f_q = s_q$ $\forall q \in V$. Note in $\underline{S}_{pre}$, the restriction map is the identity so $f_x = f_y$ $\forall x, y \in V$ $\Rightarrow$ s_q are the same for all $q \in V$. $\Rightarrow$ locally constant at p.

$(s_p)_{p \in U}$ can be seen as function $U \to S$ that is locally constant at every point. $\rightsquigarrow$ $\underline{S}_{pre}^{sh}$ is $\underline{S}$, constant sheaf.

2.4.8 (sheafification for escape étalé).

2.4.9. Subsheaves and quotient sheaves

2.4N $\phi : \mathcal{F} \to \mathcal{G}$ morphism of sheaves of sets on top space X. Following are equivalent:

(a) ϕ monomorphism in category of sheaves

(b) ϕ injective on level of stalks : $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$ injective $\forall p \in X$

(c) ϕ injective on level of open sets: $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ injective for all open $U \subset X$.

Proof:

(b) $\Rightarrow$ (a) : Consider $\alpha, \beta : \mathcal{H} \to \mathcal{F}$ morphisms of sheaves s.t.

$\phi\alpha = \phi\beta. \Rightarrow \phi_p \circ \alpha_p = \phi_p \circ \beta_p \ \forall p \in X$

Since ϕ_p injective $\Rightarrow \alpha_p = \beta_p$ for all $p \in X$.

By 2.4D, we find $\alpha = \beta$. (only need $\mathcal{F}$ to be separated sheaf, $\mathcal{G}$ presheaf)

(c) $\Rightarrow$ (b) : Suppose $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ injective for all $U \subset X$.

Let $s_p, t_p \in \mathcal{F}_p$ s.t. $\phi_p(s_p) = \phi_p(t_p)$ or $\phi(s)|_V = \phi(t)|_V$ for some open $V \ni p$ $\Rightarrow \phi(s|_V) = \phi(t|_V)$ (ϕ commutes with res)

$\Rightarrow s|_V = t|_V$ since $\phi(V)$ injective $\Rightarrow s_p = t_p$ as $p \in V$.

Thus, ϕ_p is injective for all $p \in X$.

(a) $\Rightarrow$ (c): Suppose $\phi : \mathcal{F} \to \mathcal{G}$ monomorphism of sheaves.

For open $U \subset X$, we show $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ injective.

Consider sheaf $\mathcal{H}$ on X defined as follows:

$$\mathcal{H}(V) = \begin{cases} \{1,2\} & \text{if } V \subseteq U \\ \{1\} & \text{otherwise} \end{cases}$$ with obvious restriction map.

Check this is a sheaf

If $s, t \in \mathcal{F}(U)$ s.t. $\phi(s) = \phi(t)$. We define $\alpha, \beta : \mathcal{H} \to \mathcal{F}$ as follows:

- at all V, α, β send 1 to s.
- when $V \subseteq U$, $\alpha(V)$ sends 2 to $s|_V$,

UNSOLVE

2.4O Show following are equivalent $\phi : \mathcal{F} \to \mathcal{G}$

(a) ϕ epimorphism in category of sheaves

(b) ϕ surjective on level of stalks $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$ is surjective for all $p \in X$.

Proof: (b) $\Rightarrow$ (a) : If we have sheaf $\mathcal{H}$ and sheaves morphisms $\alpha, \beta : \mathcal{G} \to \mathcal{H}$ s.t. $\alpha\phi = \beta\phi \Rightarrow \alpha_p\phi_p = \beta_p\phi_p \ \forall p \in X \Rightarrow \alpha_p = \beta_p$ (ϕ_p surjective)

$\Rightarrow$ (2.4D) $\alpha = \beta$.

(a) $\Rightarrow$ (b) : Suppose ϕ epimorphism but ϕ_p not surjective for some $p \in X$. Then $\exists\, s_p \in \mathcal{G}_p$ so $s_p \notin \phi_p(\mathcal{F}_p)$. Say s_p is $s \in \mathcal{G}(V)$.

Consider skyscraper sheaf $i_{p,*}(a,b)$ defined as $i_{p,*}(U) = \begin{cases} \{a,b\} & \text{if } p \in U \\ \{e\} & p \notin U \end{cases}$

Define $\alpha, \beta : \mathcal{G} \to i_{p,*}$ as $\alpha(U)f = \begin{cases} a & \text{if } p \in U \\ e & p \notin U \end{cases}$

$$\beta(U)f = \begin{cases} b & \text{if } (f,U) \sim (s,V) \text{ at } p \in V \cap U \\ a & \text{if } (f,U) \nsim (s,V) \text{ with } p \in U \cap V \\ e & \text{if } p \notin U \end{cases}$$

Both are morphisms of sheaves and $\alpha \neq \beta$ (evaluated at s gives $a \neq b$)

And $\beta\phi = \alpha\phi$ as $\beta(U) \circ \phi(U) f = \alpha(U) \circ \phi(U) f = \begin{cases} a & \text{if } p \in U \\ e & p \notin U \end{cases}$

$\Rightarrow$ contradiction as ϕ epimorphism .

- If condition holds, then $\mathcal{G}$ quotient sheaf of $\mathcal{F}$.
- Monomorphism and Epimorphisms of sheaves can be checked at level of stalks
- Both exercises generalize readily to sheaves with values in any reasonable category where "injective" replaced by monomorphism, "surjective" replaced by epimorphism

$\boxed{2.4.10}$ $\mathbb{C} = X$ classical topology, $\mathcal{O}_X$ sheaf of holomorphic functions

$\mathcal{O}_X^*$ sheaf of invertible (nowhere zero) holomorphic functions.

This is sheaf of abelian grps under multiplication.

$$0 \to \underline{\mathbb{Z}} \xrightarrow{\times 2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1$$

- Show $\exp : \mathcal{O}_X \to \mathcal{O}_X^*$ describes $\mathcal{O}_X^*$ as a quotient sheaf of $\mathcal{O}_X$.

  It suffices to check this stalkwise. First, we compute $\mathcal{O}_{X,p}$ and $\mathcal{O}_{X,p}^*$.

  Consider holomorphic function on $p \in U$. We know f coincides with its Taylor series at p. ... $\mathcal{O}_{X,p} \simeq \mathbb{C}\{x\}$ $\mathcal{O}_{X,p}^* \simeq \mathbb{C}\{x\}^\times$

- At $U = \mathbb{C}^\times$, map exp is not surjective as $z \mapsto z$ in $\mathcal{O}_X^*(\mathbb{C}^\times)$ but it does not have logarithm.

§2.5. Recovering sheaves from "sheaf on a base".

$\boxed{2.5.A}$ Fix sheaf $\mathcal{F}$ on topological space X, with base $\{B_i\}$ of open sets on X. Then the information $(\{\mathcal{F}(B_i)\}, \{res_{B_i,B_j} : \mathcal{F}(B_i) \to \mathcal{F}(B_j)\})$ is enough to recover $\mathcal{F}$.

Recover $\mathcal{F}(U)$: This can be identified with compatible germs. And we know how to do this with information given

Recover restriction map $res_{U,V}$: also do this stalkwise

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, hook] \arrow[d, "res_{U,V}"'] & \prod_{p\in U} \mathcal{F}_p \arrow[d] \\
\mathcal{F}(V) \arrow[r, hook] & \prod_{p\in V} \mathcal{F}_p
\end{tikzcd}$$

$\boxed{2.5.1}$. Suppose $\{B_i\}$ base on X, F is a sheaf of sets on this base. Then there is sheaf $\mathcal{F}$ extending F (with isomorphism $\mathcal{F}(B_i) \cong F(B_i)$ agreeing with the restriction maps). This sheaf $\mathcal{F}$ is unique up to unique isomorphism

Proof: Define $\mathcal{F}$ as sheaf of compatible germs of F.

Define stalk of presheaf F on base $p \in X$ by $F_p = \varinjlim F(B_i)$. the colimit is over all B_i containing p.

Define $\mathcal{F}(U) = \{(f_p \in F_p)_{p\in U}$; for all $p \in U$, exists B with $p \in B \subset U$
$s \in F(B)$ with $s_q = f_q$ for all $q \in B\}$.

This is a sheaf (same reason why compatible germs are sheaf 2.4.I).

$\boxed{2.5B}$ $F(B) \to \mathcal{F}(B)$ is an isomorphism.

$f \in F(B)$ sends to $(f_p \in F_p)_{p\in B}$

- Injective: Given $f, g \in F(B)$ so $(f_p \in F_p)_{p\in B} = (g_p \in F_p)_{p\in B}$.
  $\Rightarrow f_p = g_p \ \forall p \in B \Rightarrow f|_{B_p} = g|_{B_p}$ for some base $B_p \ni p$.
  By base identity axiom, we have $f = g$
- Surjective: Given $(f_p \in F_p)_{p\in B}$ in $\mathcal{F}(B)$.
  - By definition, for each $p \in B$, we find $s \in F(B')$ s.t. $s_q = f_q$ for all $q \in B'$.
  - Consider two such $s_1 \in F(B_1)$, $s_2 \in F(B_2)$ and $B_2 \cap B_1 \neq \phi$.
    For any base $B_3 \subset B_1 \cap B_2$, we need to show $s_1|_{B_3} = s_2|_{B_3}$.
    Since $s_{1p} = s_{2p} = f_p$ for all $p \in B_3$, there exists open $B' \subset B_3$ so $s_1|_{B'} = s_2|_{B'}$. As this is true for all $p \in B_3$, union of such B' is $B_3 \Rightarrow$ by identity axiom $s_1|_{B_3} = s_2|_{B_3}$, as desired. Thus, for any $B_3 \subset B_1 \cap B_2$ then $s_1|_{B_3} = s_2|_{B_3}$
    $\Rightarrow$ By base gluability, exists $s \in F(B)$ so $s|_{B_i} = s_i$.
    and $s_p = f_p \ \forall p \in B$ ┘

We show $\mathcal{F}$ is unique up to unique isomorphism, i.e. if there are two sheaves $\mathcal{F}, \mathcal{G}$ extending F then $\mathcal{F} \simeq \mathcal{G}$ where the isomorphism is unique.

For any B_i, as $\mathcal{F}(B_i) \simeq F(B_i) \cong \mathcal{G}(B_i)$ so $\mathcal{F} \cong \mathcal{G}$.

Show uniqueness: Suppose we have 2 different isomorphisms between $\mathcal{F}$ and $\mathcal{G}$. Then $\exists B$ base so $a \in \mathcal{F}(B)$ is sent to $b, c \in \mathcal{G}(B)$ via these two different isomorphism.

Since the isomorphisms respect restrictions so $b|_{B'} = c|_{B'}$ for any $B' \subset B$. By base identity, $b = c$. □

$\boxed{2.5C}$ (Morphisms of sheaves corresponding to morphisms of sheaves on base)

$\{B_i\}$ base of topology of X. A morphism $F \to G$ of sheaves on base is collection of maps $F(B_k) \to G(B_k)$ s.t.

$$\begin{tikzcd}
F(B_i) \arrow[r] \arrow[d, "res_{B_i,B_j}"'] & G(B_i) \arrow[d, "res_{B_i,B_j}"] \\
F(B_j) \arrow[r] & G(B_j)
\end{tikzcd}$$

commutes $\forall B_j \hookrightarrow B_i$

(a) Give morphism of sheaves $\phi: \mathcal{F} \to \mathcal{G}$.

The induced morphism of sheaves on base is just $\overset{\overset{F(B_i)}{\|}}{\mathcal{F}(B_i)} \to \overset{\overset{G(B_i)}{\|}}{\mathcal{G}(B_i)}$

(b) Given $\phi: F \to G$ morphism of sheaves on base $\{B_i\}$.

Morphism $\psi: \mathcal{F} \to \mathcal{G}$ between the induced sheaves is defined as:

$$\psi(f_p \in F_p)_{p\in U} = (\phi(f)_p \in G_p)_{p\in U}.$$

- We first show $(\phi(f)_p \in G_p)_{p\in U}$ is in $\mathcal{G}(U)$. For every $p \in U$, there exists base $B \subset U$ and $s \in F(B)$ so $s_q = f_q$ for all $q \in B$.
  $\Rightarrow s|_{B'} = f|_{B'}$ for some $q \in B' \subset B \Rightarrow \phi(s)|_{B'} = \phi(s|_{B'}) = \phi(f|_{B'}) = \phi(f)|_{B'}$
  $\Rightarrow \phi(s)_q = \phi(f)_q \ \forall q \in B'$, as desired.
- Show the ψ commutes with the restriction map is clear. □

$\boxed{2.5.2}$ Equivalence of categories between:

$\mathcal{A} = \{\text{Sheaves on given base}\}$ $\quad$ $\{\text{Sheaves on } X\} = \mathcal{B}$

Φ: $\quad F(U) \mapsto \mathcal{F}(U) = \{(f_p \in F_p)_{p\in U} \text{ compatible}\}$.

$(\alpha: F \to G) \longmapsto \beta: \mathcal{F} \to \mathcal{G}$ $\quad \beta((f_p \in F_p)_{p\in U}) = (\alpha(f)_p \in G_p)_{p\in U}$

Ψ: $\quad F(B) = \mathcal{F}(B) \longleftarrow\!\shortmid \ \mathcal{F}(U)$

$\alpha: \mathcal{F}(B) \to \mathcal{G}(B) \longleftarrow\!\shortmid \ \beta: \mathcal{F} \to \mathcal{G}$

Check $\Psi\circ\Phi \simeq id_{\mathcal{A}}$:

$$\begin{tikzcd}
(\Psi\circ\Phi)(F) \arrow[r, "(\Psi\circ\Phi)(\alpha)"] \arrow[d, "m_F"'] & (\Psi\circ\Phi)(G) \arrow[d, "m_G"] \\
id_{\mathcal{A}}(F) \arrow[r, "id_{\mathcal{A}}(\alpha)"] & id_{\mathcal{A}}(G)
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
(\Psi\circ\Phi)(F)(B) \arrow[r, "(\Psi\circ\Phi)(\alpha)"] \arrow[d, "m_F"'] & (\Psi\circ\Phi)(G)(B) \arrow[d, "m_G"] \\
F(B) \arrow[r, "\alpha(u)"] & G(B)
\end{tikzcd}$$

$$\Downarrow$$

Thus, m_F, m_G isomorphisms in this case from 2.5B. $\leftsquigarrow$

$$\begin{tikzcd}
\{(f_p \in F_p)_{p\in B} \text{ compatible}\} \arrow[r] \arrow[d, "m_F"'] & \{(\alpha(f)_p \in G_p)_{p\in B} \text{ compatible}\} \arrow[d, "m_G"] \\
F(B) \arrow[r, "\alpha"] & G(B)
\end{tikzcd}$$

$\boxed{2.5.3}$ - $\mathcal{O}_X$-module F on base $\{B_i\}$ $\quad F(B_i)$ is $\mathcal{O}_X(B_i)$-module

- Given $\mathcal{O}_X$-module F on base $\{B_i\}$, the corresponding sheaf $\mathcal{F}(U) = \{(f_p \in F_p)_{p\in U} \ldots\}$ is $\mathcal{O}_X(X)$-module as F_p is $\mathcal{O}_{X,p}$-module.

$$\begin{matrix} (s_p \in \mathcal{O}_{X,p})_{p\in U} \circlearrowright \mathcal{F}(U) \\ \wr\wr \\ \mathcal{O}_X(U) \end{matrix}$$

2.5D Suppose $X = \cup u_i$ is open cover of X, and we have sheaves $\mathcal{F}_i$ on U_i along with isomorphisms (of sheaves) $\phi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$ (with ϕ_{ii} the identity) that agree on triple overlaps, i.e. $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ on $U_i \cap U_j \cap U_k$. (this is called cocycle condition). Show that these sheaves can be glued together into sheaf on X (unique upto unique isomorphism), s.t. $\mathcal{F}_i \cong \mathcal{F}|_{U_i}$, and the isomorphisms over $U_i \cap U_j$ are obvious ones.

Proof: - If $\{B_i\}$ is a base of topological space X, we can refine this base by asking that for every B_i must contained in some U_j. (the result is still a base, as for any $p \in X$ and open $U \ni p$, exist basis element B so $p \in B \subset U$).

- We define our sheaf $\mathcal{F}$ on X by constructing its sheaf F on base $\{B_i\}$ by: for base B, exists $U_i \supset B$ where U_i in open cover $\{U_i\}$ of X. then we define $F(B) = \{(f_p \in \mathcal{F}_{i,p})_{p \in B}$ compatible$\}$
  - define $res_{B,B'} : F(B) \to F(B')$ as follows. Suppose we define $F(B)$ wrt $B \subset U_i$ and $F(B')$ wrt $B' \subset U_j$. then
    $res_{B,B'} : (f_p \in \mathcal{F}_{i,p})_{p \in B} \longmapsto (\phi_{ij}(f)_p \in \mathcal{F}_{j,p})_{p \in B'}$
    (note $U_i \supset B \supset B'$, $B' \subset U_j$)
- Check that it's a presheaf: $B_i \subset B_j \subset B_k$ then $res_{B_k B_i} = res_{B_j B_i} res_{B_k B_j}$ (This is where we use the cocycle condition).
  Suppose $F(B_i), F(B_j), F(B_k)$ are defined wrt $B_\ell \subset U_\ell$.
  then $res_{B_k B_i} = res_{B_j B_i} res_{B_k B_j}$ amounts to $\phi_{ki} = \phi_{ji} \circ \phi_{kj}$ on $B_i \subset U_i \cap U_j \cap U_k$.
- Check base identity: $U_\ell \supset B = \cup B_i$ and $f = (f_p \in \mathcal{F}_{\ell,p})_{p \in B}$ and $g \in (g_p \in \mathcal{F}_{\ell,p})_{p \in B}$ in $F(B)$ s.t. $res_{B,B_i} f = res_{B,B_i} g$ for all i. This is equivalent to $\phi_{\ell i}(f)_p = \phi_{\ell,i}(g)_p \ \forall p \in B_i$
  $\Rightarrow f_p = g_p \ \forall p \in B_i$ as $\phi_{\ell i}$ isomorphism of sheaves (2.4E).
  $\Rightarrow f = g$.
- Check base gluability: $B = \cup B_i$ $f_i = (f_{i,p} \in \mathcal{F}_{i,p})_{p \in B_i} \in F(B_i)$ s.t. $res_{B_i B_k} f_i = res_{B_j B_k} f_j \ \forall B_k \subset B_i \cap B_j$, i.e. $\phi_{ik}(f_i)_p = \phi_{jk}(f_j)_p$ for all $p \in B_k$.
  Suppose $B \subset U_\ell$. Because $\phi_{\ell,i}$ isomorphism so its stalk map also isomorphism, meaning for every $f_{i,p} \in \mathcal{F}_{i,p}$ then exists $f_{\ell,p} \in \mathcal{F}_{\ell,p}$ so $\phi_{\ell i}(f_\ell)_p = f_{i,p}$. $(p \in B_i \subset U_i \cap U_\ell)$. Because of cocycle condition and isomorphism of ϕ_{ij}, $f_{\ell,p}$ is well-defined for all $p \in B$ (i.e. if $\phi_{ik}(f_i)_p = \phi_{jk}(f_j)_p$ then preimages of $f_{i,p}$ and $f_{j,p}$ via $\phi_{\ell,i}$ and $\phi_{\ell,j}$ are the same).
  Define $f = (f_{\ell,p} \in \mathcal{F}_{\ell,p})_{p \in B} \in F(B)$. We find $f|_{B_i} = f_i$.
- Thus, we find F is a sheaf on base $\{B_i\}$. Its corresponding sheaf is denoted $\mathcal{F}$. To show $\mathcal{F}_i \cong \mathcal{F}|_{U_i}$ for all i, it suffices to show $F|_{B_i} \cong F_i$ (where F_i is sheave on base of $\mathcal{F}_i$).
- Check unique up to unique isomorphism: If we have two sheaves $\mathcal{F}$ and $\mathcal{G}$ s.t. $\mathcal{F}|_{U_i} \cong \mathcal{F}_i \cong \mathcal{G}|_{U_i}$ $\Leftrightarrow$ $F|_B \cong F_i|_B \cong G|_B$ on base.
  $\Rightarrow F \cong G$ on base The morphism is unique because it respects restriction map,

2.5.E Suppose morphism of sheaves $F \to G$ on base $\{B_i\}$ is surjective for all B_i ($F(B_i) \to G(B_i)$ surjective for all i). Show that the corresponding morphism of sheaves is epimorphism. [i.e. morphism of sheaves is epimorphism if it is surjective on a base].

Proof: Corresponding sheaf $\mathcal{F}$ of F is determined by $F(U) \ni f \mapsto (f_p \in F_p)_{p \in U} \in \mathcal{F}(U)$ with $\phi : F \to G$ and $\psi : \mathcal{F} \to \mathcal{G}$

If $\mathcal{H}$ sheaf and $\alpha, \beta : \mathcal{G} \to \mathcal{H}$ morphisms of sheaves s.t. $\alpha\psi = \beta\psi$. We need to show $\alpha = \beta$. Consider $(g_p \in G_p)_{p \in B} \in \mathcal{G}(B)$. By exercise 2.5B, $\exists g \in G(B)$ s.t. g has that as compatible germ. As ϕ surjective so $\exists f \in F(B)$ so $\phi(f) = g \Rightarrow (\phi(f)_p \in G_p)_{p \in B} = (g_p \in G_p)_{p \in B} \Rightarrow \alpha(g) = (\alpha \circ \psi)(f) = (\beta \circ \psi)(f) = \beta(g) \ \forall g \in \mathcal{G}(B)$
$\Rightarrow \alpha(B) = \beta(B)$ for all base $B \Rightarrow \alpha = \beta$.
$\Rightarrow \psi : \mathcal{F} \to \mathcal{G}$ is epimorphism.

- Try this with example 2.4.10. UNSOLVE.

§2.6. Sheaves of abelian groups, and $\mathcal{O}_X$-modules, form an abelian categories

Category of sheaves of abelian groups on topological space X is an additive category. (the 0 object here is $0(U) = \{0\}$ for all $U \subset X$). To show that it's an abelian category, we must show first that any morphism $\phi: \mathcal{F} \to \mathcal{G}$ has kernel and cokernel.

$\boxed{2.6.A}$ Stalk of the kernel is the kernel of the stalks: for all $p \in X$, there is natural isomorphism $(\ker(\mathcal{F} \to \mathcal{G}))_p \cong \ker(\mathcal{F}_p \to \mathcal{G}_p)$.

Let $\phi: \mathcal{F} \to \mathcal{G}$. We have exact sequence

$$0 \to \ker\phi \xrightarrow{i} \mathcal{F} \xrightarrow{\phi} \mathcal{G}. \quad \text{i.e. } i \text{ monomorphism}$$

$\Rightarrow i_p$ injective $\Rightarrow (\ker\phi)_p \cong i(\ker\phi)_p$. And we show that $i(\ker\phi)_p$ is kernel of $\mathcal{F}_p \to \mathcal{G}_p$.

- Because of exactness at $\mathcal{F}$ so $\phi i = 0$ so $\phi_p i_p = 0 \Rightarrow \operatorname{im} i_p \subset \ker\phi_p$.
- If $f_p \in \ker\phi_p$, then $\phi(f)_p = 0$ or $\phi(f) = 0 \in \mathcal{G}(U)$.
  As $0 \to (\ker\phi)(U) \xrightarrow{i} \mathcal{F}(U) \xrightarrow{\phi} \mathcal{G}(U)$ exact so $\exists g \in (\ker\phi)(U)$ so $i(g) = f$
  $\Rightarrow i_p(g_p) = f_p \Rightarrow f_p \in (\operatorname{im} i_p)$

$\Rightarrow \operatorname{im} i_p = i(\ker\phi)_p$ is kernel of ϕ_p.

- "Natural" isomorphism here refers to the fact that if $(\ker\phi, i)$ with $\ker\phi \xrightarrow{i} \mathcal{F}$ is the kernel of morphism $\mathcal{F} \xrightarrow{\phi} \mathcal{G}$ of sheaves then $((\ker\phi)_p, i_p)$ is kernel of $\mathcal{F}_p \xrightarrow{\phi_p} \mathcal{G}_p$ (note the i here)

$\boxed{2.6.1}$ (Cokernel of sheaves) Consider sheaves morphism $\phi: \mathcal{F} \to \mathcal{G}$ which has cokernel $\mathcal{H}_{pre}$ in category of presheaves. Let $sh: \mathcal{H}_{pre} \to \mathcal{H}$ be its sheafification. Then the composition $\mathcal{G} \to \mathcal{H}_{pre} \to \mathcal{H}$ is cokernel of ϕ in category of sheaves.

$\boxed{2.6\,B}$ Stalk of cokernel is naturally isomorphic to cokernel of the stalk.

$$(\operatorname{coker}\phi)_p \cong \operatorname{coker}\phi_p \qquad \phi: \mathcal{F} \to \mathcal{G}.$$

(essentially just to show $\mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{s} \operatorname{coker}\phi \to 0$ being exact will imply exactness at level of stalk).

We will show that $(\operatorname{coker}\phi)_p \cong \operatorname{coker}\phi_p \cong \operatorname{im}(s_p)$.

- Since s epimorphism so s_p surjective for all $p \in P$. This follows $\operatorname{im}(s_p) \cong (\operatorname{coker}\phi)_p$
- If we know exactness at $\mathcal{G}_p$ of $\mathcal{F}_p \xrightarrow{\phi_p} \mathcal{G}_p \xrightarrow{s_p} (\operatorname{coker}\phi)_p \to 0$ then $\operatorname{im}\phi_p = \ker s_p$. We know $\operatorname{coker}\phi_p = \mathcal{G}_p / \operatorname{im}\phi_p = \mathcal{G}_p / \ker s_p \cong \operatorname{im}(s_p)$, hence $(\operatorname{coker}\phi)_p \cong \operatorname{im}(s_p) \cong \operatorname{coker}\phi_p$, as desired
- Thus, it suffices to show exactness at $\mathcal{G}_p$.
  Since $\mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{s} \operatorname{coker}\phi$ is exact so $s\phi = 0 \Rightarrow s_p \phi_p = 0$.
  Next, we need $\ker s_p \subset \operatorname{im}\phi_p$. Let $f_p \in \ker(s_p) \Rightarrow s(f)_p = 0$
  $\Rightarrow s(f) = 0 \in \mathcal{G}(U)$. Due to exactness at $\mathcal{F}(U) \xrightarrow{\phi} \mathcal{G}(U) \xrightarrow{s} (\operatorname{coker}\phi)(U)$ exists $g \in \mathcal{F}(U)$ so $\phi(g) = f \Rightarrow \phi_p(g_p) = f_p$
  $\Rightarrow \ker s_p \subset \operatorname{im}\phi_p$
  We find $\ker s_p = \operatorname{im}\phi_p$, as desired.
- This is a natural isomorphism, with the same reason as in 2.6A ┘

⊛ Kernel and cokernel may be checked at the level of stalks: This is the content of two exercises 2.6A, 2.6B. To be more precise: $(\mathcal{H}, i)$ is kernel of $\phi: \mathcal{F} \to \mathcal{G}$ iff $(\mathcal{H}_p, i_p)$ is kernel of ϕ_p for all $p \in X$.

- 2.6A proved "$\Rightarrow$" direction. For the "$\Leftarrow$" direction. Suppose $(\mathcal{H}_p, i_p)$ is kernel of ϕ_p for all $p \in X$. This follows $\phi_p i_p = 0$ or $(\phi i)_p = 0\ \forall p \Rightarrow \phi i = 0$ (2.4D).
  Hence, $\exists! f: \mathcal{H} \to \ker\phi$ s.t. $i = \alpha f$.

$$\begin{tikzcd}
\ker\phi \arrow[r, "\alpha"] & \mathcal{F} \arrow[r, "\phi"] & \mathcal{G} \\
\mathcal{H} \arrow[u, dashed, "f"] \arrow[ur, "i"] \arrow[urr, "0"'] & &
\end{tikzcd}$$

- Since $(\mathcal{H}_p, i_p)$ kernel of $\mathcal{F}_p \xrightarrow{\phi_p} \mathcal{G}_p$, which is also $((\ker\phi)_p, \alpha_p)$ so we find $\mathcal{H}_p \cong (\ker\phi)_p$ via unique isomorphism $f': \mathcal{H}_p \to (\ker\phi)_p$ so $i_p = \alpha_p f_p$. This follows $f' = f_p$.
  $\Rightarrow f_p$ is isomorphism for all $p \Rightarrow f$ isomorphism (2.4E).
  $\Rightarrow (\mathcal{H}, i)$ is kernel of $\mathcal{F} \xrightarrow{\phi} \mathcal{G}$. ▨

$\boxed{2.6C}$ Suppose $\phi: \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of abelian groups. Show that image sheaf $\operatorname{im}\phi$ is the sheafification of the image presheaf. Show that the stalk of the image is the image of the stalk.

- We show $sh(\ker(\operatorname{coker}_{pre}\phi)) = sh(\operatorname{im}_{pre}\phi)$ is image sheaf of ϕ:

$$\begin{tikzcd}
 & \mathcal{F} \arrow[r] & \mathcal{G} \arrow[r, "s"] & \operatorname{coker}_{pre}\phi \arrow[r, "sh"] & \operatorname{coker}\phi \\
\mathcal{H} \arrow[urr, "f"] \arrow[r, dashed, "g"'] & \operatorname{im}_{pre}\phi \arrow[ur, "i"] \arrow[r, "sh"'] & \operatorname{im}\phi \arrow[u, dashed, "\exists!\psi"'] & &
\end{tikzcd}$$

⊕ As $\operatorname{im}_{pre}\phi \xrightarrow{i} \mathcal{G}$, exists unique ψ from $\operatorname{im}\phi$ (sheafification of $\operatorname{im}_{pre}\phi$) to $\mathcal{G}$ so $i = \psi\, sh$. We note that ψ is monomorphism. Indeed, this is equivalent to ψ_p being injective for all p. Note that $i_p = \psi_p sh_p$ and sh_p is an isomorphism by 2.4M and i_p injective (as i is kernel of s) $\Rightarrow \psi_p$ injective for all p, as desired.

⊕ Consider $\mathcal{H} \xrightarrow{f} \mathcal{G}$ s.t. $\mathcal{H} \xrightarrow{f} \mathcal{G} \to \operatorname{coker}\phi$ is 0 then $sh \circ s \circ f = 0 \Rightarrow s \circ f = 0$ (from definition of sh). $\Rightarrow$ by universal prop of $\operatorname{im}_{pre}\phi$ as kernel of s, $\exists! g: \mathcal{H} \to \operatorname{im}_{pre}\phi$ s.t. $f = ig$.
$\Rightarrow$ This induces map $sh \circ g: \mathcal{H} \to \operatorname{im}\phi$ s.t. $f = \psi \circ (sh \circ g)$.
We show that this map $\mathcal{H} \to \operatorname{im}\phi$ that factors f through $\operatorname{im}\phi$ is unique.
Indeed, if $\exists h: \mathcal{H} \to \operatorname{im}\phi$ s.t. $f = \psi h \Rightarrow \psi h = \psi(sh \circ g)$
As ψ is monomorphism so $h = sh \circ g$, as desired.

$\Rightarrow$ Thus, $\operatorname{im}\phi$ is indeed kernel of $\mathcal{G} \to \operatorname{coker}\phi$, making it image sheaf of $\phi: \mathcal{F} \to \mathcal{G}$.

- We show stalk of image is image of stalk: $(\operatorname{im}\phi)_p \cong \operatorname{im}(\phi_p)$.

$$\begin{aligned}
(\operatorname{im}\phi)_p &\cong \ker((\operatorname{coker}\phi)_p) && \text{(as kernel of stalk is stalk of kernel)} \\
&\cong \ker(\operatorname{coker}(\phi_p)) && \text{(as cokernel of stalk is stalk of cokernel)} \\
&\cong \operatorname{im}\phi_p.
\end{aligned}$$
┘

⊛ Exactness of a sequence of sheaves may be checked at the level of stalks. (this is also proved in 2.6A, 2.6B).

2.6D Taking the stalk of a sheaf of abelian groups is an exact functor.
i.e. if X topological space and $p \in X$, show taking stalk at p defines exact functor $Ab_X \to Ab$. (already prove exactness at 2.6A, 2.6B).

2.6E $X = \mathbb{C}$ classical topology. Define $\mathcal{O}_X$ sheaf of holomorphic functions, $\mathcal{O}_X^*$ sheaf of invertible (nowhere zero) holomorphic functions. $\underline{\mathbb{Z}}$ constant sheaf associated to $\mathbb{Z}$. We have exact seq of sheaves:

$$0 \to \underline{\mathbb{Z}} \xrightarrow{\times 2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1$$

Since kernel, image, cokernel are determined at the level of stalks, it suffices to check that

$$0 \to \underline{\mathbb{Z}}_p \xrightarrow{\times 2\pi i} \mathcal{O}_{X,p} \xrightarrow{\exp} \mathcal{O}_{X,p}^* \to 1$$

is exact seq of abelian groups, i.e.

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathbb{C}\{z\} \xrightarrow{\exp} \mathbb{C}\{z\}^\times \to 1$$

2.6F Left-exactness of functor of "sections over U":
- Suppose $U \subset X$ open, $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ exact seq of sheaves of abelian grps. Show $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$ exact.

Since "forgetful" right adjoint to sheafification functor is left-exact so apply it to $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$, we obtain exact seq of presheaves. But evaluation at U is just the same as $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$ so this seq is exact.

- Exponential exact seq shows that is section functor need not be exact.

2.6G (Left-exactness of push forward) Suppose $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$ is an exact sequence of sheaves of abelian groups on X. If $\pi : X \to Y$ continuous map, show

$$0 \to \pi_* \mathcal{F} \xrightarrow{\alpha_*} \pi_* \mathcal{G} \xrightarrow{\beta_*} \pi_* \mathcal{H} \qquad (1)$$

is exact (The previous exercise, dealing with left-exactness of global section functor can be interpreted as special case of this, in the case where Y is 1 point).
- Recall $\pi_* \mathcal{F}(U) = \mathcal{F}(\pi^{-1}(U))$ for $U \subset Y$ open.

$(\pi_*\mathcal{F})_p$ is then (f, V) where $f \in \mathcal{F}(\pi^{-1}(V))$ and $(f,V) \sim (g,U)$ when $\exists W \subset V \cap W$ s.t. $f|_{\pi^{-1}(W)} = g|_{\pi^{-1}(W)}$. $p \in V \subset Y$

It suffices to check left-exactness of (1) stalkwise, which is not difficult.

2.6H (Left-exactness of $\mathcal{H}om$) Suppose $\mathcal{F}$ is a sheaf of abelian groups on topological space X. Show $\mathcal{H}om_{Ab_X}(\mathcal{F}, \cdot)$ is left-exact covariant functor $Ab_X \to Ab_X$
Show $\mathcal{H}om(\cdot, \mathcal{F})$ left-exact contravariant functor.

Suppose $0 \to \mathcal{G} \xrightarrow{\alpha} \mathcal{H} \xrightarrow{\beta} \mathcal{E}$ exact seq of sheaves of abelian groups. We have

$$(*) \qquad 0 \to \mathcal{G}_p \xrightarrow{\alpha_p} \mathcal{H}_p \xrightarrow{\beta_p} \mathcal{E}_p \text{ exact.}$$

We need to show

$$(1) \qquad 0 \to \mathcal{H}om(\mathcal{F}, \mathcal{G})_p \to \mathcal{H}om(\mathcal{F}, \mathcal{H})_p \to \mathcal{H}om(\mathcal{F}, \mathcal{E})_p \text{ exact}$$

- Recall $\mathcal{H}om_{Ab_X}(\mathcal{F}, \mathcal{G})(U) := Mor(\mathcal{F}|_U, \mathcal{G}|_U)$.

$\mathcal{H}om_{Ab_X}(\mathcal{F}, \mathcal{G})_p$ can be identified with group hom $Hom(\mathcal{F}_p, \mathcal{G}_p)$
(but not the other way around)

- To show (1) is exact, it suffices to show

$$0 \to Hom_{Ab_X}(\mathcal{F}, \mathcal{G})_p \xrightarrow{\alpha_{*,p}} Hom_{Ab_X}(\mathcal{F}, \mathcal{H})_p \xrightarrow{\beta_{*,p}} Hom_{Ab_X}(\mathcal{F}, \mathcal{E})_p$$

is exact.

- Show $\alpha_{*,p}$ is monomorphism: Let $f_p \in Hom_{Ab_X}(\mathcal{F}, \mathcal{G})_p$ then $\alpha_{*,p}(f_p)$ is $(\alpha \circ f)_p = \alpha_p \circ f_p$. So if $\alpha_{*,p}(f_p) = 0$ then $f_p = 0$ as α_p injective.
- As $\beta_{*,p} \circ \alpha_{*,p} = \beta_p \circ \alpha_p = 0$ so $\operatorname{Im} \alpha_{*,p} \subseteq \ker \beta_{*,p}$.
- We need $\ker \beta_{*,p} \subseteq \operatorname{Im} \alpha_{*,p}$. Consider $f_p \in Hom_{Ab_X}(\mathcal{F}, \mathcal{H})_p$ so $\beta_{*,p}(f_p) = \beta_p \circ f_p = 0$. $\Rightarrow$ $\beta \circ f = 0$ over $\mathcal{F}|_U$ for some open U.
Note we know $0 \to \mathcal{H}om(\mathcal{F}|_U, \mathcal{G}|_U) \to \mathcal{H}om(\mathcal{F}|_U, \mathcal{H}|_U) \to \mathcal{H}om(\mathcal{F}|_U, \mathcal{E}|_U)$
exact so $\exists g : \mathcal{F}|_U \to \mathcal{G}|_U$ so $\alpha \circ g = f$.
$\Rightarrow \alpha_p \circ g_p = f_p \Rightarrow f_p \in \operatorname{Im} \alpha_{*,p}$.

(*) **Remark:** Proof of 2.6F, 2.6G, 2.6H are all the same.
left exact $\Leftrightarrow$ left-exact at stalk $\Rightarrow$ this can be obtained by lifting to level of open sets.
Essentially 2.6G, 2.6H don't have full short exactness because 2.6F fails in the first place.

2.6I $(X, \mathcal{O}_X)$ ringed space. Then $\mathcal{O}_X$-module form an abelian category.
Sketch: - Check $\mathcal{O}_X$-module form additive cat: how to define product of two $\mathcal{O}_X$-modules
- Check that mono, epic, ker, coker, exact of $\mathcal{O}_X$ modules can be checked at level of stalks

2.6J
- Categorical def of tensor product of $\mathcal{O}_X$-modules:

$\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. Define $\mathcal{O}_X$-bilinear map $\phi : \mathcal{F} \times \mathcal{G} \to \mathcal{H}$ between 2 $\mathcal{O}_X$-modules to be set of $\mathcal{O}_X(U)$-bilinear $\phi(U) : \mathcal{F}(U) \times \mathcal{G}(U) \to \mathcal{H}(U)$
i.e. $r \in \mathcal{O}_X(U)$, $x \in \mathcal{F}(U)$, $y \in \mathcal{G}(U)$ $\quad \phi(rx, y) = \phi(x, ry)$,
$\phi(x+y, z) = \phi(x,z) + \phi(y,z)$, $\phi(x, y+z) = \phi(x,y) + \phi(x,z)$
so that ~~it~~ is consistent with the restriction map, i.e. it is morphism of sheaves. (it seems no need for $\mathcal{O}_X$-module structure to preserve).

Then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is $\mathcal{O}_X$-module and $\mathcal{O}_X$-bilinear $\phi : \mathcal{F} \times \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is universal wrt to all other $\mathcal{H}$ and $\mathcal{O}_X$-bilinear $f. \mathcal{F} \times \mathcal{G} \to \mathcal{H}$.

- Construction. Define presheaf tensor product of $\mathcal{O}_X$ modules $\mathcal{F}, \mathcal{G}$ as

$$(\mathcal{F} \otimes_{pre} \mathcal{G})(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X} \mathcal{G}(U)$$

then take its sheafification.

Chapter 3: Toward affine scheme

§3.1. Toward schemes

- A "geometric space":
- will define scheme to be following data
  - set
  - topology
  - structured sheaf: sheaf of ring.
- Key example of scheme: complex (affine) varieties

3.1.1 Differentiable manifolds: (Note: everything Vakil said here is informal! There will be notion of locally ringed space making this rigorous!)

X differentiable manifolds ← it is topological space
has sheaf of differentiable functions $\mathcal{O}_X$

- Can define a differentiable real manifolds as topological space X with a sheaf of rings. Require a cover of X by open sets s.t. on each open set the ringed space $\mathcal{O}_X(U)$ is isomorphic to a ball around origin in $\mathbb{R}^n$ with the sheaf of differentiable functions on that ball.

With this def, the ball is the basic patch, and a general manifold is obtained by glueing the patches together.

- The basic patch is the notion of an affine scheme.
- Differentiable maps of differentiable manifolds $\pi : X \to Y$. is a continuous map so that we can pull back differentiable func on Y along this continuous map to get differentiable functions on X.

More formally, we have map $\pi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$

Inverse image is left-adjoint to pushforward, so we also get a map $\pi^{\#} : \mathcal{O}_Y \to \pi_* \mathcal{O}_X$.

3.1.A Suppose that $\pi : X \to Y$ is continuous map of differentiable manifolds (Just a continuous map) Show π is differentiable if differentiable functions pull back to differentiable functions, i.e. if pull back by π gives a map $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$.

For $U \subset Y$ open then $(\pi_* \mathcal{O}_X)(U) = \mathcal{O}_X(\pi^{-1}(U))$. Hence, the map $\mathcal{O}_Y(U) \to (\pi_* \mathcal{O}_X)(U)$ is $f \in \mathcal{O}_Y(U) \mapsto f \circ \pi$. Thus, the map indicates that if f is differentiable on open set of Y then $f \circ \pi$ is differentiable on open set of X. This is the (usual) definition of differentiable map.

3.1.B Morphism of differentiable manifolds $\pi : X \to Y$ with $\pi(p) = q$ induces morphism of stalks $\pi^{\#} : \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$. Show that $\pi^{\#}(m_{Y,q}) \subset m_{X,p}$.
In other words, if you pullback a function that vanishes at q, you get a function that vanishes at p.

- Morphism of differentiable manifolds $\pi : X \to Y$ induces map $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$ sending $f \in \mathcal{O}_Y(U) \mapsto f \circ \pi \in \pi_* \mathcal{O}_X(U)$. This induces $\mathcal{O}_{Y,q} \to (\pi_* \mathcal{O}_X)_q$

By 2.2.I, we have $(\pi_* \mathcal{O}_X)_q \to \mathcal{O}_{X,p}$ by restricting $f \in \mathcal{F}(\pi^{-1}(U))$ to $\mathcal{F}(V)$ where $p \in V \subset \pi^{-1}(U)$.

Thus, we obtain morphism of stalks $\pi^{\#} : \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$

- We show $\pi^{\#}(m_{Y,q}) \subset m_{X,p}$.

If we view $m_{Y,q}$ as functions vanishing at q then obviously $\pi^{\#}(m_{Y,q})$ contains functions vanishing at p from the description of $\pi^{\#}$.

3.1.2 π induces map on tangent spaces

$$(m_{X,p}/m_{X,p}^2)^\vee \to (m_{Y,q}/m_{Y,q}^2)^\vee$$

This is clear as $\pi^{\#}(m_{Y,q}) \subset m_{X,p}$ so $\pi^{\#}(m_{Y,q}^2) \subset m_{X,p}^2$.

§3.2. The underlying set of affine schemes.

3.2.1 For ring A, denote $\operatorname{Spec} A$ set of prime ideals of A. Denote prime ideal $\mathfrak{p}$ of A as element of $\operatorname{Spec} A$ as $[\mathfrak{p}]$.

3.2.3 Some examples:

Example 1 (complex affine line) $\mathbb{A}^1_{\mathbb{C}} := \operatorname{Spec} \mathbb{C}[x]$. We find prime ideals of $\mathbb{C}[x]$. As $\mathbb{C}[x]$ integral domain, 0 is prime. Also, $(x-a)$ is prime, for all $a \in \mathbb{C}$: it is even a maximal ideal, as quotient by this ideal is a field:

$$0 \to (x-a) \longrightarrow \mathbb{C}[x] \xrightarrow{f \mapsto f(a)} \mathbb{C} \longrightarrow 0 .$$

⋮

3.2.A (a) Describe the set $\operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$. Ring $k[\varepsilon]/(\varepsilon^2)$ called ring of dual numbers.

0 is not prime ideal of $k[\varepsilon]/(\varepsilon^2)$ as $\varepsilon \cdot \varepsilon = 0$.

Let $\mathfrak{p}$ be prime ideal of $k[\varepsilon]/(\varepsilon^2)$. Then if $\alpha \neq 0$ in $\mathfrak{p}$ then $1 \in \mathfrak{p}$, contradiction. If $\alpha\varepsilon \neq 0$ in $\mathfrak{p}$ then $\varepsilon \in \mathfrak{p}$ so $(\varepsilon) \subseteq \mathfrak{p}$. If $\alpha + \beta\varepsilon \in \mathfrak{p}$ with $\beta \neq 0, \alpha \neq 0$ then $\varepsilon(\alpha+\beta\varepsilon) \in \mathfrak{p} \Rightarrow \alpha\varepsilon \in \mathfrak{p} \Rightarrow \varepsilon \in \mathfrak{p} \Rightarrow (\varepsilon) \subseteq \mathfrak{p}$.

We show $\mathfrak{p} \subseteq (\varepsilon)$. If $\alpha + \beta\varepsilon \in \mathfrak{p}$ with $\alpha \neq 0$ then we find $\varepsilon \in \mathfrak{p}$ as above $\Rightarrow \alpha \in \mathfrak{p} \Rightarrow 1 \in \mathfrak{p}$, contradiction. Thus, $\operatorname{Spec} k[\varepsilon]/(\varepsilon^2) = \{(\varepsilon)\}$

(b) $\operatorname{Spec} k[x]_{(x)} = \{(0), (x)\}$

We will prove more general result (3.2.K):

Suppose S is a multiplicative subset of A. There is an order-preserving bijection of prime ideals of $S^{-1}A$ with the prime ideals of A that don't meet the multiplicative set S.

$= \phi^{-1}(\mathfrak{p})$ where $\phi : A \to S^{-1}A$

Proof: Consider prime ideal $\mathfrak{p}$ of $S^{-1}A$. Let I be set of $a \in A$ s.t. $a/s \in \mathfrak{p}$ for some $s \in S$. Then I is an ideal of A and $I \subset \mathfrak{p}$. Since $\mathfrak{p}$ is prime ideal of $S^{-1}A$ so I prime ideal of A that don't meet S.

Conversely, if I prime ideal of A, let $\mathfrak{p} = \left\{\frac{a}{s} : a \in I, s \in S\right\}$. Then $\mathfrak{p}$ is ideal of $S^{-1}A$. If $\frac{a}{s} = \frac{b}{s_1} \cdot \frac{c}{s_2}$ then $s_3(s_1 s_2 a - sbc) = 0$ for some $s \in S$. so $s_3 sbc \in I$ as $a \in I$. Note $I \cap S = \phi$ so $s_3 s \notin I \Rightarrow bc \in I$ (as I prime) $\Rightarrow b \in I$ or $c \in I$. Thus $b/s_1 \in \mathfrak{p}$ or $c/s_2 \in \mathfrak{p}$. We find $\mathfrak{p}$ prime ideal of $S^{-1}A$. ∎

- Example 5 (the affine line over $\mathbb{R}$) $\mathbb{A}^1_{\mathbb{R}} = \operatorname{Spec} \mathbb{R}[x]$. $\mathbb{R}[x]$ is Euclidean domain, hence PIDs, hence UFDs. Also, irreducible polynomial in $\mathbb{R}[x]$ must have degree ≤ 2. Indeed, consider $f \in \mathbb{R}[x]$, over $\mathbb{C}[x]$, we can write $f(x) = (x-\lambda_1) \cdots (x-\lambda_n)$ with $n = \deg(f) \geq 3$. Note $0 = \overline{f(\lambda_i)} = f(\overline{\lambda_i})$ since coef of f are real. Thus, if $\lambda_i \in \mathbb{C} \setminus \mathbb{R}$ root of f then $\lambda_k = \overline{\lambda_i}$ also root of f. We also have $(x-\lambda_i)(x-\lambda_k) = x^2 - (\overline{\lambda_i} + \lambda_i)x + \lambda_i \overline{\lambda_i} \in \mathbb{R}[x]$. Thus, f factorises into real polynomials of degree 1 and 2.

With this, we find that prime ideals of $\mathbb{R}[x]$ are (0), $(x-a)$ where $a \in \mathbb{R}$ and (x^2+ax+b) where x^2+ax+b is irreducible quadratic. The latter two are maximal ideals as $\mathbb{R}[x]/(x-a) \cong \mathbb{R}$ via $\varphi : \mathbb{R}[x] \to \mathbb{R}$ sending $f \mapsto f(a)$. and $\mathbb{R}[x]/(x^2+ax+b) \cong \mathbb{C}$ via $\varphi : \mathbb{R}[x] \to \mathbb{R}$ sending $f \mapsto f\left(\frac{-a-\sqrt{a^2-4b}}{2}\right)$

3.2.B

$\Rightarrow$ $\mathbb{A}^1_{\mathbb{R}}$ pictured as complex plane, folded along the real axis (i.e. we consider Galois-conjugate points $a \pm ib$ $(b \neq 0)$ as one).

3.2C Describe the set $\mathbb{A}^1_{\mathbb{Q}}$.

Not explicit description but elements in $\mathbb{A}^1_{\mathbb{Q}}$ corresponds to irreducible polynomials in $\mathbb{Q}[x]$ (as $\mathbb{Q}[x]$ PID, meaning (a) prime iff a irreducible).
$\Rightarrow$ View this as points of $\mathbb{A}^1_{\overline{\mathbb{Q}}}$ where Galois-conjugates are glued together.

Example 6 (affine line over $\mathbb{F}_p$) $\mathbb{A}^1_{\mathbb{F}_p} = \operatorname{Spec} \mathbb{F}_p[x]$. As, $\mathbb{F}_p[x]$ Euclidean domain, prime ideals of the form (0) or $(f(x))$ where $f(x)$ irreducible polynomials, which can be of any degree. Irreducible polynomials correspond to sets of Galois conjugates in $\overline{\mathbb{F}_p}$ (i.e. elements sharing same minimal polynomial over $\mathbb{F}_p$).

3.2D Spec $k[x]$ has infinitely many points.

Prime ideal in $k[x]$ is principal ideal generated by irreducible elements.
If has finite prime ideals, i.e. finite # irreducible elements $p_1, \dots, p_n$ then $f = p_1 \cdots p_n + 1$ is irreducible, hence (f) is prime.

Example 7 (complex affine plane) $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,y]$. $\mathbb{C}[x,y]$ is not principal ideal domain as (x,y) is not principal ideal.

All prime ideals of $\mathbb{A}^2_{\mathbb{C}}$ are: 1) (0) 2) $(x-a, y-b)$ 3) $(f(x,y))$ for irreducible polynomial $f(x,y)$.

3.2E Show these are all the prime ideals of $\mathbb{C}[x,y]$.

Suppose $\mathfrak{p}$ prime ideal of $\mathbb{C}[x,y]$ that is not principal.
Choose $f \in \mathfrak{p}$ s.t. f is irreducible (exists as choosing the element with smallest deg in $\mathfrak{p}$)
Then $(f) \subset \mathfrak{p}$. Since $\mathfrak{p}$ not principal, $\exists g \in \mathfrak{p}$ s.t. $f \nmid g$. $\Rightarrow f, g$ coprime.
Use Euclidean algorithm in Euclidean domain $\mathbb{C}(y)[x]$, $\exists A, B \in \mathbb{C}[x,y]$, $h \in \mathbb{C}[y]$ s.t. $Af + Bg = h \Rightarrow h \in \mathfrak{p}$. Since $\mathbb{C}$ alg closed, can write h as product of $y-b \Rightarrow \exists\, y-b \in \mathfrak{p}$ as $\mathfrak{p}$ is prime
Similarly, $\exists x-a \in \mathfrak{p} \Rightarrow (x-a, y-b) \subseteq \mathfrak{p}$. But $(x-a, y-b)$ maximal so $\mathfrak{p} = (x-a, y-b)$.

Q: What does the parabola $y = x^2$ represent in diagram 3.3?

Locus of points (i.e. prime ideals) vanishing at function $y - x^2 \in \mathbb{C}[x,y]$

3.2F For maximal ideal $\mathfrak{m}$ of $k[x_1, \dots, x_n]$ then $k[x_1, \dots, x_n]/\mathfrak{m}$ is a field, in particular a finite (hence algebraic) extension of k. As, k is algebraically closed, the residue field must be k.

Now, we can identify $x_i + \mathfrak{m}$ with $a_i + \mathfrak{m}$ so $x_i - a_i \in \mathfrak{m}$ for some $a_i \in k$.
$\overset{\text{ideal}}{}$
Hence $(x_1 - a_1, \dots, x_n - a_n) \subseteq \mathfrak{m}$. But $(x_1 - a_1, \dots, x_n - a_n)$ maximal so this is $\mathfrak{m}$.

3.2G An integral domain A which is finite k-algebra (a k-algebra that is finite dimensional vector space over k) must be a field.

Indeed, consider multiplication by $x \neq 0$. $f : A \to A$. This is a linear map
It is injective as A is integral domain. Since it is finite dimensional so $\times x$ is vector space isomorphism. Thus, f has inverse g. We have $f(g(1)) = 1$ so $x g(1) = 1 \Rightarrow x$ has inverse.

Example 8 (complex affine n-space) $\mathbb{A}^n_{\mathbb{C}} := \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]$.
Consider the case $n=3$. We list few elements of $\mathbb{A}^3_{\mathbb{C}}$:

- "0-dimensional": $(x-a, y-b, z-c)$ are the only maximal ideals of $\mathbb{C}[x_1, \dots, x_n]$.
  We picture this as point (a,b,c) in $\mathbb{C}^3$.
- "1-dimensional": (x,y) as $\mathbb{C}[x,y,z]/(x,y) \cong \mathbb{C}[z]$ is integral domain. (Interpret as functions on the z-axis). We picture (x,y) as locus where $x=y=0$, which is z-axis. (later, we know "1-dim" prime ideals correspond to irreducible curves)
- "2-dimensional": $(f(x,y,z))$ where $f(x,y,z)$ is irreducible, associated with "hypersurface" $f=0$, which is "2-dimensional".
- "3-dimensional": (0). is everywhere but nowhere in particular.

3.2H Describe the maximal ideal of $\mathbb{Q}[x,y]$ corresponding to $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$. It is $(x-y, xy-2)$. We show $\mathbb{Q}[x,y]/(x-y, x^2-2)$ is a field
With $(x-y)$, we view elements of $F = \mathbb{Q}[x,y]/(x-y, x^2-2)$ as pol over x.
With (x^2-2), elements of F viewed as $ax+b$ where $a, b \in \mathbb{Q}$.
Note $a_1x + b_1 \neq a_2x + b_2$ for $(a_1, b_1) \neq (a_2, b_2)$ so $F \cong \mathbb{Q}^2$.
- Similarly for $(-\sqrt{2}, \sqrt{2})$ and $(\sqrt{2}, -\sqrt{2})$ which we have $(x+y, x^2-2)$.

§3.2.6 Quotients and localizations

$\boxed{3.2J}$ A ring, I ideal of A and $\phi: A \to A/I$. Show ϕ^{-1} gives inclusion preserving bijection between prime ideals of A/I and prime ideals of A containing I.

If $\mathfrak{p}$ prime ideal of A containing I, $\phi(\mathfrak{p}) =: \mathfrak{p}+I$ is a prime ideal: if $(a+I)(b+I) \in \mathfrak{p}+I$ then $ab = p+i$ for $p \in \mathfrak{p}$, $i \in I \Rightarrow ab \in \mathfrak{p}$ (as $I \subset \mathfrak{p}$) so either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

If $\mathfrak{p}$ prime ideal of A/I, $\phi^{-1}(\mathfrak{p}) = \{p+i : \phi(p) \in \mathfrak{p}, i \in I\}$.
It is prime ideal: if $ab \in \phi^{-1}(\mathfrak{p}) \Rightarrow \phi(a)\phi(b) \in \mathfrak{p} \Rightarrow$ either $a \in \phi^{-1}(\mathfrak{p})$ or $b \in \phi^{-1}(\mathfrak{p})$.

$\Rightarrow$ View Spec A/I as a subset of Spec A.

$\Rightarrow$ Affine complex varieties. Suppose A is finitely generated $\mathbb{C}$-algebra, generated by $x_1, \ldots, x_n$, with relations $f_1(x_1, \ldots, x_n) = \ldots = f_r(x_1, \ldots, x_n) = 0$. This gives us an interpretation of Spec A as a subset of $\mathbb{A}^n_{\mathbb{C}}$, which we think as "traditional points" (n-tuples of complex numbers, corresponding to prime ideal (0) of A, i.e. solutions to $f_1 = \ldots = f_n = 0$) and "bonus" points we haven't fully described

$\boxed{3.2K}$ (see 3.2A) View Spec $S^{-1}A$ as subset of Spec A as prime ideals of $S^{-1}A$ in bijection with prime ideals of A that don't meet S.

- There are two important localizations: $A_f = \{1, f, f^2, \ldots\}^{-1}A$ where $f \in A$ with motivating example $A = \mathbb{C}[x,y]$, $f = y - x^2$; the second is $A_{\mathfrak{p}} = (A-\mathfrak{p})^{-1}A$, where $\mathfrak{p}$ is a prime ideal with motivating example $A = \mathbb{C}[x,y]$, $S = A - (x,y)$.
- If $S = \{1, f, f^2, \ldots\}$ then prime ideals of $S^{-1}A$ are just those prime ideals (in A) not containing f — the points where "f doesn't vanish". So to picture Spec $\mathbb{C}[x,y]_{y-x^2}$, we picture the affine plane, and throw out those points on the parabola $y-x^2$, i.e. points (a, a^2) for $a \in \mathbb{C}$ (by which we mean $[(x-a, y-a^2)]$ as $y - x^2 = y - a^2 - (x-a)(x+a) = 0 \mod (x-a, y-a^2)$) as well as the "new kind of points" $[(y-x^2)]$.
- It can be confusing to think about localization in the case where zerodivisors are inverted, because localization $A \to S^{-1}A$ is not injective. Geometric intuition can help: Consider $A = \mathbb{C}[x,y]/(xy)$ and $f = x$. What is localization A_f? Space $\mathbb{C}[x,y]/(xy)$ "is" union of two axes in the plane (elements in $\mathbb{C}[x,y]/(x,y)$ can be viewed as $x \cdot f(x) + g(y)$ or $f(x) + y g(y)$ where $f(x) \in \mathbb{C}[x]$, $g(y) \in \mathbb{C}[y]$; hence $(x, y-a)$ and $(x-a, y)$ for $a \in \mathbb{C}$ are its maximal ideals). Localizing means throwing out the locus where x vanishes. So we are left with x-axis, minus the origin, so we expect Spec $\mathbb{C}[x]_x$. So there should be natural isomorphism

$$(\mathbb{C}[x,y]/(x,y))_x \xrightarrow{\sim} \mathbb{C}[x]_x.$$

LHS has $y=0$ as $\frac{0}{x} = y$.

$\boxed{3.2.I}$

$$\begin{tikzcd}
\mathbb{C}[x,y]/(x,y) \arrow[r, "\alpha"] \arrow[d, shift left, "g'"] & (\mathbb{C}[x,y]/(x,y))_x \arrow[d, dashed, shift left, "\exists g"] \\
\mathbb{C}[x] \arrow[r, "\beta"'] \arrow[u, shift left, "f'"] & \mathbb{C}[x]_x \arrow[u, dashed, shift left, "\exists f"]
\end{tikzcd}$$

- In order to construct map f from $\mathbb{C}[x]_x$ to $(\mathbb{C}[x,y]/(x,y))_x$ we use universal property of $\mathbb{C}[x]_x$, we want $f': \mathbb{C}[x] \to \mathbb{C}[x,y]/(x,y)$ s.t. $\alpha f'$ sending $\{1, x, x^2, \ldots\}$ to invertible elements in $(\mathbb{C}[x,y]/(x,y))_x$.
  $\Rightarrow f'(x^i) = x^i + (x,y)$
  This induces f s.t. $f\beta = \alpha f'$. (1)
- Similarly define $g': \mathbb{C}[x,y]/(x,y) \to \mathbb{C}[x]$ s.t. $y \mapsto 0$, i.e.
  $f(x) + y g(y) \mapsto f(x) + g(0)$.
  This induces g s.t. $g\alpha = \beta g'$ (2)
- Note f', g' are isomorphism with $f'g' = 1$, $g'f' = 1$. (3)
- (1), (2), (3) $\Rightarrow g\alpha f' = \beta$, $f\beta g' = \alpha \Rightarrow gf\beta = \beta$ and $fg\alpha = \alpha$.
- By universal prop of $\mathbb{C}[x]_x$ applied to gf, we find $gf = 1$
  Similarly, $fg = 1$.
  ⌟

- If $S = A - \mathfrak{p}$, prime ideals of $S^{-1}A$ are just prime ideals of A contained in $\mathfrak{p}$. In our example, $A = \mathbb{C}[x,y]$, $\mathfrak{p} = (x,y)$ we keep all those points corresponding to "things through the origin", i.e. the 0-dimensional point (x,y), the 2-dim point (0), and those 1-dim points $(f(x,y))$ where $f(0,0) = 0$. → think of this being a shred of the plane near the origin; anything not actually "visible" at the origin is discarded.
- When $A = K[x]$, $\mathfrak{p} = (x)$ (or more generally when $\mathfrak{p}$ is maximal ideal). Then $A_{\mathfrak{p}}$ has only 2 prime ideals $([(0)], [(x)])$.

3.2.9. Maps of rings induce maps of spectra (as sets).

$\boxed{3.2M}$ $\phi : B \to A$ ring map, $\mathfrak{p}$ prime ideal in A, $\phi^{-1}(\mathfrak{p})$ prime ideal of B

Hence map of rings $\phi : B \to A$ induces maps of sets $\operatorname{Spec} A \to \operatorname{Spec} B$. i.e. contravariant functor from cat of rings to cat of sets.

$\boxed{3.2N}$ (Reality check) Look at the proofs 3.2K, 3.2J.

$\boxed{3.2.10}$ Example of "affine complex varieties".

– A parabola $b = a^2$ in $\mathbb{C}$, viewed as locus of points (prime ideals) s.t. $b - a^2$ vanishes at these point, i.e. $b - a^2$ lies in these points / prime ideals $\rightsquigarrow \operatorname{Spec} \mathbb{C}[a,b]/(b-a^2)$

– Similarly, a "curve" in $\mathbb{C}^3$ cut out by equations $y = x^2$, $z = y^2$ corresponds to $\operatorname{Spec} \mathbb{C}[x,y,z]/(y-x^2, z-y^2)$

– The map from the parabola $b = a^2$ sends $(a,b) \in \mathbb{C}^2$ to the point $(a,b,b^2) \in \mathbb{C}^3$ in the curve. We have a map:

$$\operatorname{Spec} \mathbb{C}[a,b]/(b-a^2) \longrightarrow \operatorname{Spec} \mathbb{C}[x,y,z]/(y-x^2, z-y^2).$$

given by

$$\begin{tikzcd}
\mathbb{C}[a,b]/(b-a^2) & \mathbb{C}[x,y,z]/(y-x^2, z-y^2) \arrow[l] \\
(a,b,b^2) & (x,y,z) \arrow[l, mapsto]
\end{tikzcd}$$

$\boxed{3.2P}$ (a) A, B rings. Let $\phi : B \to A$ be ring morphism. Then ϕ induces map of sets $\operatorname{Spec} A/I \to \operatorname{Spec} B/J$ for any ideals $I \subset A$ and $J \subset B$ s.t. $\phi(J) \subset I$.

Since $\phi(J) \subset I$ so this induces $B/J \to A/I$, which then induces $\operatorname{Spec} A/I \to \operatorname{Spec} B/J$ from 3.2M

(b) Apply (a) for $B = k[y_1, \ldots, y_n]$, $A = k[x_1, \ldots, x_m]$ where $\phi : B \to A$ sends $y_i \mapsto f_i$ for some $f_i \in k[x_1, \ldots, x_m]$. Then it induces map of sets $\operatorname{Spec} k[x_1, \ldots, x_m]/I \to \operatorname{Spec} k[y_1, \ldots, y_n]/J$ for $\phi(J) \subset I$.

For $I = J = 0$: consider maximal ideal $(x_1 - a_1, \ldots, x_m - a_m)$ in $k[x_1, \ldots, x_m]$.

Note $\phi(y_i - f_i(a_1, \ldots, a_m)) = f_i(x_1, \ldots, x_m) - f_i(a_1, \ldots, a_m) = \phi_i(x_1, \ldots, x_m)$

As $\phi_i(a_1, \ldots, a_m) = 0 \Rightarrow \phi_i(x_1, \ldots, x_m) = \sum (x_1 - a_1)^{h_1} \cdots (x_m - a_m)^{h_m} \in (x_1 - a_1, \ldots, x_n - a_m)$

$$\Longrightarrow y_i - f_i(a_1, \ldots, a_m) \in \phi^{-1}(x_1 - a_1, \ldots, x_m - a_m)$$

$$\Rightarrow \underbrace{(y_1 - f_1(a_1, \ldots, a_m), \ldots, y_n - f_n(a_1, \ldots, a_m))}_{\text{maximal ideal}} \underset{\overset{\Downarrow}{=}}{\subseteq} \phi^{-1}(\underbrace{(x_1 - a_1, \ldots, x_m - a_m)}_{\text{prime}})$$

Thus, $\operatorname{Spec} k[x_1, \ldots, x_m] \to \operatorname{Spec} k[y_1, \ldots, y_n]$ sends $(a_1, \ldots, a_m)$ to $(f_1(a_1, \ldots, a_m), \ldots, f_n(a_1, \ldots, a_m)) \in k^n$.

$\boxed{3.2Q}$ (Picturing $\mathbb{A}^n_{\mathbb{Z}}$) Consider map of sets $\pi : \mathbb{A}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$, given by the ring map $\mathbb{Z} \to \mathbb{Z}[x_1, \ldots, x_n]$. If p is prime, there is a bijection between fiber $\pi^{-1}([(p)])$ and $\mathbb{A}^n_{\mathbb{F}_p}$.

- We determine $\pi^{-1}([(p)])$ as follows: The ring map $\mathbb{Z} \to \mathbb{Z}[x_1, \ldots, x_n]$ is $a \mapsto a$ and $\pi : I \in \mathbb{A}^n_{\mathbb{Z}} \mapsto \pi^{-1}(I)$. Hence if $\pi^{-1}(I) = [(p)]$ then $I \subset \mathbb{Z}[x_1, \ldots, x_n]$ must contain p and no other primes of $\mathbb{Z}$. $\Rightarrow \pi^{-1}([(p)])$ are primes ideals of $\mathbb{Z}[x_1, \ldots, x_n]$ containing prime p but not other primes in $\mathbb{Z}$.

- If $f \in \mathbb{Z}[x_1, \ldots, x_n]$, $f(x_1, \ldots, x_n) = \sum_{i_1, \ldots i_n} a_{i_1, \ldots i_n} x_1^{i_1} \cdots x_n^{i_n}$ then let $f_p \in \mathbb{F}_p[x_1, \ldots, x_n]$ by $f_p(x_1, \ldots, x_n) = \sum_{i_1, \ldots i_n} (a_{i_1, \ldots i_n} \bmod p) x_1^{i_1} \cdots x_n^{i_n}$.

Consider $I \in \pi^{-1}([p])$, let $I^p = \{ f_p \in \mathbb{F}_p[x_1, \ldots, x_n] \mid f \in I \}$.

This is an ideal of $\mathbb{F}_p[x_1, \ldots, x_n]$ as $p \in I$. Furthermore, if I is prime then I_p is prime (as $(fg)_p = f_p g_p + pC$ for some $C \in \mathbb{Z}[x_1, \ldots, x_n]$).

- Conversely, consider $I^p \in \mathbb{A}^n_{\mathbb{F}_p}$, let $I = \{ i + pk : i \in I^p, k \in \mathbb{Z}[x_1, \ldots, x_n] \}$

$\Rightarrow$ Bijection $\pi^{-1}([(p)])$ with $\mathbb{A}^n_{\mathbb{F}_p}$.

- Fiber over $[(0)]$: $\pi^{-1}([(0)])$ is in bijection with $\mathbb{A}^n_{\mathbb{Q}}$.

Indeed, $\pi^{-1}([(0)])$ contains prime ideals of $\mathbb{A}^n_{\mathbb{Z}}$ that intersect trivially with $\mathbb{Z}$.

$I \in \pi^{-1}([(0)])$ then I also prime in $\mathbb{Q}[x_1, \ldots, x_n]$. Conversely, if I prime in $\mathbb{Q}[x_1, \ldots, x_n]$ then $I \cap \mathbb{Z} = \{0\}$ and let $I' = \{ nf \mid f \in I, n \in \mathbb{Z} \}$ (sufficiently large n s.t. $nf \in \mathbb{Z}[x_1, \ldots, x_n]$ and gcd coef of nf is 1).

Then I' ideal of $\mathbb{Z}[x_1, \ldots, x_n]$. It is prime.

$\Rightarrow$ Think $\mathbb{A}^n_{\mathbb{Z}}$ as "$\mathbb{A}^n$-bundle" over $\operatorname{Spec} \mathbb{Z}$.

3.2.11 Ring $k[\varepsilon]/(\varepsilon^2)$ has unique prime ideal (ε).

Two functions ε and 0 both vanish at (ε) but they are not the same

$\Rightarrow$ Functions are not determined by their values at points

$\Rightarrow$ fault of nilpotents

3.2R (a) If I ideal of nilpotents, then inclusion $\operatorname{Spec} B/I \to \operatorname{Spec} B$ is bijection. $\rightsquigarrow$ Nilpotents don't affect the underlying set of scheme

(b) Show nilpotents of a ring B form an ideal. This ideal called nilradical and is denoted $\mathfrak{N} = \mathfrak{N}(B)$.

Proof: (a) Because every prime ideal contains nilpotents.

(b) $(f+g)^m = \sum f^n g^{m-n}$ is 0 for f, g nilpotents and for large m. ⌟

3.2.12. Theorem: The nilradical $\mathfrak{N}(A)$ is intersection of all prime ideals of A.

Geometrically: Function on $\operatorname{Spec} A$ vanishes at every point $\Leftrightarrow$ it is nilpotent.

3.2S Proof of above theorem.

32R shows $\mathfrak{N}(A)$ lies inside all prime ideals.

For other direction, if $x \notin \mathfrak{N}(A)$ we wish to show there is a prime ideal not containing x. Consider A_x, which is not 0-ring, so there exists prime ideal (or maximal ideal by Zorn lemma) $\rightsquigarrow$ corresponds to prime ideal of A not containing x.

- If $\mathfrak{N} = (0)$ then functions are determined by their values at points.

If ring has no nonzero nilpotents, we say it is reduced

§3.4. The underlying topological space of an affine scheme

→ Zariski topology:

- If S subset of ring A, define Vanishing set of S by
$$V(S) = \{[\mathfrak{p}] \in \operatorname{Spec} A : S \subset \mathfrak{p}\}.$$

It is set of points on which all elements of S are zero. We define this to be closed subsets.

Eg: $V(xy, yz) \subset \mathbb{A}^3_{\mathbb{C}}$. then points in $V(xy,yz)$ are solutions to $xy = yz = 0$ (which is not just ordinary points but also $(y), (x,z), \dots$)

- We define the Zariski topology to be $V(S)$ is closed for all ideals S of A. Need to check it's a topology.

$\boxed{3.4C}$ (a) $\phi = V(0)$ and $\operatorname{Spec} A = V(1)$ so they are both open subsets of $\operatorname{Spec} A$

(b) If I_i collection of ideals, then $\cap_i V(I_i) = V(\sum_i I_i)$.

Consider $[\mathfrak{p}] \in \cap_i V(I_i) \Leftrightarrow I_i \subset \mathfrak{p}\ \forall i \Leftrightarrow \sum_i I_i \subset \mathfrak{p}$. $\Rightarrow$..

$\Rightarrow$ union of collection of open sets is open

(c) Show $V(I_1) \cup V(I_2) = V(I_1 I_2)$.

$[\mathfrak{p}] \in V(I_1) \cup V(I_2) \Longleftrightarrow$ either $I_1 \subset \mathfrak{p}$ or $I_2 \subset \mathfrak{p} \Leftrightarrow I_1 I_2 \subset \mathfrak{p}$.

$\Rightarrow$ intersection of finite # open sets is open. ┘

- Func $V(\cdot)$ is inclusing reversing: If $S_1 \subset S_2$ then $V(S_1) \supset V(S_2)$.

$\boxed{3.4D}$ If $I \subset A$ ideal then define its radical by
$$\sqrt{I} = \{r \in A : r^n \in I \text{ for some } n \in \mathbb{Z}^{>0}\}.$$

For example, the nil radical $\mathfrak{N}$ is $\sqrt{(0)}$. Then $\sqrt{I}$ an ideal (3.2R(b)) and $V(I) = V(\sqrt{I})$. We say an ideal is radical if it equals its own radical. Show $\sqrt{\sqrt{I}} = \sqrt{I}$ and prime ideals are radicals.

Proof: - Show $V(I) = V(\sqrt{I})$. Since $I \subset \sqrt{I}$ so $V(\sqrt{I}) \subset V(I)$. If $\mathfrak{p} \in V(I)$ then $I \subset \mathfrak{p}$. Since $\mathfrak{p}$ is prime ideals, if $r^n \in I \subset \mathfrak{p}$ so $r \in \mathfrak{p} \Rightarrow \sqrt{I} \subset \mathfrak{p}$.

- Two corollaries:
  - As $(I \cap J)^2 \subset IJ \subset I \cap J$ and $V(I \cap J) = V((I \cap J)^2)$ as $\mathfrak{p} \in V((I \cap J)^2) \Leftrightarrow (I \cap J)^2 \subset \mathfrak{p} \Leftrightarrow I \cap J \subset \mathfrak{p} \Leftrightarrow \mathfrak{p} \in V(I \cap J)$
  $\Rightarrow V(IJ) = V(I \cap J)$ $(= V(I) \cup V(J)$ from 3.4C(c)$)$.
  - $V(S) = V((S)) = V(\sqrt{(S)})$. ┘

3.4.E $I_1, \dots, I_n$ ideals of ring A, Show $\sqrt{\bigcap_{i=1}^{n} I_n} = \bigcap_{i=1}^{n} \sqrt{I_i}$.

If $x \in \sqrt{\bigcap_{i=1}^{n} I_i} \Rightarrow x^k \in \bigcap_{i=1}^{n} I_i \Rightarrow x^k \in I_i \ \forall i \Rightarrow x \in \bigcap_{i=1}^{n} \sqrt{I_i}$.

If $x \in \bigcap_{i=1}^{n} \sqrt{I_i} \Rightarrow \exists k$ s.t. $x^k \in \bigcap_{i=1}^{n} I_i$ (since finitely many I_i).

$\Rightarrow x \in \sqrt{\bigcap_{i=1}^{n} I_i}$.

3.4. $\sqrt{I}$ = intersection of all prime ideals containing I.

$\subseteq$ easy

Consider A/I. The nilradical of A/I is $\sqrt{I} + I$ as $x \in A/I$, $x^k = 0$ means $x^k \in I$ or $x \in \sqrt{I}$.

By theo 3.2.12 $\Rightarrow$ $\sqrt{I} + I$ = intersection all prime ideals of A/I

$\sqrt{I}$ = ____________________ A

containing I.

3.4.2 $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x] = \{[0]\} \cup \mathbb{C} \to [(x-z)]$ $z \in \mathbb{C}$.

- Closed subsets: $V(0) = \mathbb{A}^1_{\mathbb{C}}$. As $\mathbb{C}[x]$ is PID, any ideal I is principal ideal $I = (f)$ so any other closed set

$V(f) = \{$ finite set of points $[(x-z)] \}$ $\to$ as f has finite many zero.

$\Rightarrow$ open sets: empty set, $\mathbb{A}^1_{\mathbb{C}}$ minus finite number of maximal ideals.

$\Rightarrow$ every neighborhood of every points contains $[0]$.

3.4.G Describe topological space $\mathbb{A}^1_k$. Same as 3.1.2.

open sets: empty set, $\mathbb{A}^1_k$ minus finite number of prime ideals.

3.4.3 $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,y]$ points $\leftarrow$

- $[(0)]$ 2-dim
- $(x-a, y-b)$ 0-dim
- $[(f(x,y))]$ one dim.

Closed subsets $\leftarrow$

- $V(0) = \mathbb{A}^2_{\mathbb{C}}$
- finite number of "zero dim" points
- finite # "curves" = "1-dim" points and "0-dim" points lying on it,

$\boxed{3.4.4}$ Maps of rings induces continuous maps of topological spaces.

$\boxed{3.4.H}$ $\phi : B \to A$ induces $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$ send I to $\phi^{-1}(I)$.

Show π continuous.

$V(S)$ closed in Spec B where S ideal of B. then $\pi^{-1}(V(S)) = V(\phi(S))$.

which is closed in Spec A.

$$\{[\mathfrak{p}] \in \operatorname{Spec} A \mid \underline{\phi^{-1}(\mathfrak{p}) \in V(S)}\}$$

$$\phi^{-1}(\mathfrak{p}) \in V(S) \Leftrightarrow S \subset \phi^{-1}(\mathfrak{p}) \Leftrightarrow \phi(S) \subset \mathfrak{p}$$

$\boxed{3.4.I}$ $I, S \subset B$ Ideal and multiplicative set of B.

(a) - Spec B/I naturally closed subset of Spec B.

Recall, we identify Spec B/I with subsets of Spec B as prime ideals of B containing I. (since every ideal of B/I is of J/I with $I \subset J$).

This is $V(I) = \{\mathfrak{p} \in \operatorname{Spec} B \mid I \subset \mathfrak{p}\}$.

- If $S = \{1, f, f^2, \ldots\}$ $(f \in B)$ show that Spec $S^{-1}B$ open subset of Spec B

Recall, we identify Spec $S^{-1}B$ as subsets of Spec B of all prime ideals that do not intersect S (If $I \in \operatorname{Spec} S^{-1}B$ then $I \cap B \in \operatorname{Spec} B$ that does not meet S).

$\Rightarrow$ Spec $S^{-1}B =$ Spec $B \setminus V(f)$. is open

- For arbitrary S, Spec $S^{-1}B$ need not be open or closed.

$S = \mathbb{Z}_{>0}$, $B = \mathbb{Z}$ then $S^{-1}B = \mathbb{Q}$.

We know Spec $\mathbb{Q} \subset$ Spec $\mathbb{Z}$ $\quad \{\phi, \operatorname{Spec} \mathbb{Z} - \text{finite} \# [(p)]\}$. ($p$ prime.)

$\quad\quad\quad\quad\quad\quad\quad$ as open sets of Spec $\mathbb{Z}$

$\{[(0)]\}$ is not open and not closed.

(b) Zariski topology on Spec B/I (resp. Spec $S^{-1}B$) is subspace topology induced by inclusion in Spec B.

- Closed subsets of Spec B/I.

$$V(J/I) = \{K/I \in \operatorname{Spec} B/I \text{ s.t. } J/I \subset K/I \text{ or } J \subset K\}.$$

- Closed subsets of Spec $S^{-1}B$.

$$V(I) = \{J \in \operatorname{Spec} S^{-1}B \text{ s.t. } I \subset J\} \Leftrightarrow \begin{matrix} I \cap B \subset J \cap B \\ (J \cap B) \cap S = \phi \end{matrix}$$

3.4J $I \subset B$ ideal. $f \in B$ vanishes on $V(I) \Leftrightarrow f \in \sqrt{I}$.

f vanishes on $V(I) \Longleftrightarrow f \in$ every prime of $B \supset I$.

$\Leftrightarrow f \in \sqrt{I}$ (by 3.4F).

3.4K $\operatorname{Spec} k[x]_{(x)}$ has open sets: ϕ, $\operatorname{Spec} k[x]$ minus finite number of prime ideals of $k[x]$ (including (x)).

$\{[(x)]\}$ $\parallel$

This is because $\operatorname{Spec} k[x]_{(x)}$ identify with open set $\operatorname{Spec} k[x] \setminus V(x)$ in $\operatorname{Spec} k[x] = \mathbb{A}^1_k$.

Picture $\operatorname{Spec} k[x]_{(x)}$ as $\mathbb{A}^1_k$ removing 0.

§3.5. Base of Zariski topology on Spec A

Distinguished open sets.

For $f \in A$, the distinguished open set $D(f) := \{[\mathfrak{p}] \in \operatorname{Spec} A : f \notin \mathfrak{p}\}$ is the locus where f doesn't vanish (open as $D(f) = \operatorname{Spec} A \setminus V(f)$).

3.5.A Check $D(f), f \in A$ form base for the topology.

3.5.B $f_i \in A$ for $i \in J$. Then

$$\bigcup_{i \in J} D(f_i) = \operatorname{Spec} A \underset{(1)}{\Longleftrightarrow} (\{f_i\}_{i \in J}) = A \underset{(2)}{\Longleftrightarrow} \sum_{i \in J} a_i f_i = 1 \text{ for all but finitely many } a_i \text{ being } 0.$$

- Prove (1) by contradiction:
  - If $\bigcup_{i \in J} D(f_i) \subsetneq \operatorname{Spec} A$ then exists $\mathfrak{p} \in \operatorname{Spec} A$ so $\mathfrak{p} \notin \bigcup_{i \in J} D(f_i)$ $\Rightarrow f_i \in \mathfrak{p}$ for all $i \in J \Rightarrow (\{f_i\}_{i \in J}) \subseteq \mathfrak{p}$ and $\mathfrak{p} \neq A$ so $(\{f_i\}_{i \in J}) \subsetneq A$.
  - If $(\{f_i\}_{i \in J}) \subsetneq A$ then there exists maximal ideal $\mathfrak{m} \supseteq (\{f_i\}_{i \in J})$. Implying $\mathfrak{m} \in \operatorname{Spec} A$ but $\mathfrak{m} \notin \bigcup_{i \in J} D(f_i)$.
- Prove (2). If $(\{f_i\}_{i \in J}) = A$ then $1 \in A$ can be written as $1 = \sum_{i \in J} a_i f_i$ for all but finitely many a_i being 0.

3.5C If $\bigcup_{i\in J} D(f_i) = \operatorname{Spec} A$ then from 3.5B, $\sum_{i\in J'} a_i f_i = 1$ for J' finite. $\Rightarrow (\{f_i\}_{i\in J'}) = A \Rightarrow \bigcup_{i\in J'} D(f_i) = \operatorname{Spec} A$ by 3.5B.

3.5D $D(f)\cap D(g) = D(fg)$.

3.5E $D(f)\subset D(g)$ iff $f^n\in(g)$ for some $n\geq 1$, iff g invertible element of A_f.

- If $f^n\in(g)$. Any $p\in D(f)$, i.e. $f\notin p$ then $f^n\notin p$ as p prime. But $f^n\in(g)$ so $f^n=gk$ so $g\notin p$ or $p\in D(g)$ $\Rightarrow D(f)\subseteq D(g)$.

- If $D(f)\subseteq D(g)$. then f vanishes at all points of $V(g)$. View $V(g)$ as $\operatorname{Spec} A/(g)$ then f as function on $\operatorname{Spec} A/(g)$ vanishes at all points $\Rightarrow f$ nilpotent in $A/(g)$ (theorem 3.2.12) so $f^n\in(g)$.

3.5F $D(f)=\phi \Leftrightarrow f\in p$ for all prime $p \Leftrightarrow f\in \mathfrak{N}$ radical.

§3.6 Topological and Noetherian properties

3.6.1 Connectness, Irreducibility, quasicompactness

Top space X connected $\Leftrightarrow$ not union of disjoint nonempty open sets.

3.6.A $A = A_1\times A_2\times\dots\times A_n$.

- Via induction on n, it suffices to prove this for $n=2$ as $\operatorname{Spec} A_1 \sqcup \dots \sqcup \operatorname{Spec} A_n \longrightarrow \operatorname{Spec}(A_1\times\dots\times A_{n-1}) \sqcup \operatorname{Spec} A_n \longrightarrow \operatorname{Spec} A$.

- First, we show that $\operatorname{Spec}(A_1\times A_2)$ only consists of $p_1\times A_2$ and $A_1\times p_2$ for p_1, p_2 primes in A_1, A_2.

Suppose $(a,b) = (a,1)(1,b)\in p$ prime of A.

Then either $(a,1)\in p$ or $(1,b)\in p$.

- We show that we cannot have $(a,1)$ and $(1,b)$ in p at the same time. Indeed, assume the contrary, as (a,b) also in p, we find $(0,b-1) = (a,b)-(a,1)\in p$.

$\Rightarrow (1,1) = (1,b) - (0,b-1)\in p$

$\Rightarrow p = A_1\times A_2 = A$, a contradiction.

- As this holds for any $(a,b) \in \mathfrak{p}$ (i.e. only one of $(a,1)$ or $(1,b)$ in $\mathfrak{p}$ but not both) $\Rightarrow$ $\mathfrak{p}$ is generated by (WLOG) $(a,1) \in \mathfrak{p}$. Let $\mathfrak{p}_1 = \{a \in A_1 \mid (a,1) \in \mathfrak{p}\}$ then $\mathfrak{p}_1$ is prime since $\mathfrak{p}$ is prime.

$\Rightarrow$ $\mathfrak{p}$ is generated by $(a,1)$ for $a \in \mathfrak{p}_1$.

$\Rightarrow \mathfrak{p} = \mathfrak{p}_1 \times A_2$.

Thus, primes of $A_1 \times A_2$ are either $\mathfrak{p}_1 \times A_2$ or $A_1 \times \mathfrak{p}_2$ for primes $\mathfrak{p}_1, \mathfrak{p}_2$ in A_1, A_2, respectively.

— Now, we find that $\phi_1 : \operatorname{Spec} A_1 \to D(f_1)$ sending $\mathfrak{p} \mapsto \mathfrak{p} \times A_2$ is a homeomorphism and similarly for ϕ_2. With $D(f_1) \cap D(f_2) = \phi$ so $\phi : \operatorname{Spec} A_1 \sqcup \operatorname{Spec} A_2 \to \operatorname{Spec} A$ defined via ϕ_1, ϕ_2 is a homeomorphism.

$\boxed{3.6B}$ (a) X irreducible topological space. $U \subset X$ nonempty open. If $\overline{U} \neq X$ then $X = \overline{U} \cup (X \setminus U)$, a contradiction since X irreducible $\Rightarrow \overline{U} = X$.

(b) X topological space, Z irreducible subset. then $\overline{Z}$ irreducible.

- Z irreducible means, there does not exists closed U_1, U_2 of X s.t. $Z \subset U_1 \cup U_2$ (i.e. $(U_1 \cap Z) \cup (U_2 \cap Z) = Z$) and $U_i \cap Z \subsetneq Z$.
- To show $\overline{Z}$ irreducible. Let U_1, U_2 closed of X so that $\overline{Z} \subset U_1 \cup U_2$. By irreducibility of Z, either $Z \cap U_1 = Z$ or $Z \cap U_2 = Z$. Say $Z \cap U_1 = Z$, i.e. $Z \subset U_1$. Since U_1 closed so $\overline{Z} \subset U_1 \Rightarrow \overline{Z} \cap U_1 = \overline{Z}$, as desired. Thus, $\overline{Z}$ is also irreducible.

$\boxed{3.6C}$ If A integral domain then $\operatorname{Spec} A$ irreducible.

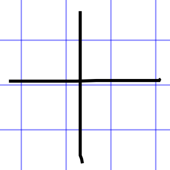
If we have $\operatorname{Spec} A = V(S_1) \cup V(S_2)$ for some closed sets $V(S_1), V(S_2)$ with S_1, S_2 ideals of A.

As $[(0)] \in \operatorname{Spec} A = V(S_1) \cup V(S_2) = V(S_1 S_2)$

$\Rightarrow S_1 S_2 \subset (0) \Rightarrow$ one of S_1, S_2 must be 0 as A an integral domain (i.e. one of $V(S_1), V(S_2)$ is $\operatorname{Spec} A$).

$\Rightarrow \operatorname{Spec} A$ is irreducible

$\boxed{3.6D}$ If X irreducible. If $X = U_1 \sqcup U_2$ for U_1, U_2 nonempty disjoint open. $\Rightarrow X = (X \setminus U_1) \cup (X \setminus U_2)$ as U_1, U_2 disjoint. Also $X \setminus U_1, X \setminus U_2$ proper closed as U_1, U_2 nonempty open.

$\boxed{3.6E}$  = solutions for $xy=0$ in $\mathbb{C}^2$ is connected but reducible as + = | union —

Solutions for $xy=0$ in $\mathbb{C}^2$

$= \text{primes in } \mathbb{C}[x,y] \text{ containing } (xy) = \operatorname{Spec} \mathbb{C}[x,y]/(xy)$.

$= V(xy)$

- It is reducible because $V(xy) = V(x) \cup V(y)$.

- It's connected. If $X = \operatorname{Spec} \mathbb{C}[x+y]/(xy)$ is not connected then by 3.6.3, there exists nonzero $a_1, a_2 \in \mathbb{C}[x,y]/(xy)$ s.t. $a_1^2 = a_1, a_2^2 = a_2, a_1 + a_2 = 1$. We show this is not possible.

- Note element in $\mathbb{C}[x,y]/(xy)$ of the form $af(x) + bg(y)$ where $a, b \in \mathbb{C}$, $f(x) \in \mathbb{C}[x]$, $g(y) \in \mathbb{C}[y]$ monic.
- And $(af(x)+bg(y))^2 = af(x) + bg(y)$ when $\deg f = \deg y = 0$ i.e. $af(x) + bg(y) = c$ constant which implies $c = 0, 1$.
- By 3.6.3, $X = \operatorname{Spec} \mathbb{C}[x,y]/(xy)$ is connected.

3.6.F (a) $I = (wz - xy, wy - x^2, xz - y^2) \subseteq k[w,x,y,z]$

Show $k[w,x,y,z]/I$ is an integral domain.

$$k[w,x,y,z]/I \simeq k[w,x,y,z,z^{-1}]/(I, z^{-1}).$$

$$= k[w,x,y,z,z^{-1}]/(w - xyz^{-1}, wy - x^2, x - y^2z^{-1}, z^{-1})$$

$$= k[xyz^{-1}, x, y, z, z^{-1}]/(xyz^{-1}\cdot y - x^2, x - y^2z^{-1}, z^{-1})$$

$$= k[(y^2z^{-1})^2 yz^{-1}, y^2z^{-1}, y, z, z^{-1}]/(z^{-1})$$

$$= k[y,z,z^{-1}]/(z^{-1}) \simeq k[y,z].$$

3.6G (a) Spec A is quasicompact.

Follows from 3.5C and the fact $D(f)$ for $f \in A$ are principal open subsets of Spec A.

3.6H (a) Topological space X that is finite union of quasicompact spaces is quasicompact.

Not hard

(b) Closed subset of quasicompact topological space is quasicompact.

Let $U \subset X$ closed in X. Let $U = \bigcup_{i \in S} U_i$ open cover of U, where $U_i = U \cap V_i$ with V_i open in X. $\Rightarrow$ $X \setminus U, V_i$ open cover of X.

$\Rightarrow$ X has finite cover $X \setminus U, V_1, \ldots, V_s$.

$\Rightarrow$ $U_1, \ldots, U_s$ open cover of U.

$\boxed{3.6.I}$ Closed points of Spec A correspond to maximal ideals. If $\mathfrak{p} \in \operatorname{Spec} A$ is closed then $\{[\mathfrak{p}]\} = V(S)$ for some ideal $S \subseteq A$.

If $\mathfrak{p} \subseteq \mathfrak{p}'$ prime then $S \subseteq \mathfrak{p} \subset \mathfrak{p}'$ so $\mathfrak{p}' \in V(S)$ so $\mathfrak{p}' = \mathfrak{p}$. Thus, $\mathfrak{p}$ is maximal.

$\boxed{3.6.G}$ By Hilbert's Nullstellensatz, closed points of $\operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_n)$ are naturally interpreted as points in $\bar{k}^n$ satisfying $f_1 = \dots = f_n = 0$.

$\boxed{3.6.J}$ (a) k field, A finitely generated k-algebra. Show closed points of Spec A are dense, by showing that if $f \in A$, $D(f)$ nonempty open set of Spec A then $D(f)$ contains closed point of Spec A.

- By 3.2.8, we identify $\operatorname{Spec} A_f$ to $D(f)$ by $\mathfrak{p}_f \in \operatorname{Spec} A_f \mapsto \mathfrak{p} = \{a \in A \mid a \in \mathfrak{p}_f\}$, i.e. via $A \overset{i}{\hookrightarrow} A_f$.
- Let $\mathfrak{p}_f$ maximal ideal of A_f. Because A finitely generated k-alg, A_f also finitely generated k-alg so by Hilbert-Nullstelensatz, $A_f/\mathfrak{p}_f$ is finite ext of k.
- Note $A \hookrightarrow A_f \to A_f/\mathfrak{p}_f$ has kernel $\mathfrak{p}$ so $A/\mathfrak{p} \simeq A_f/\mathfrak{p}_f$ is a field $\Rightarrow$ $\mathfrak{p}$ maximal ideal of A, $\mathfrak{p} \in D(f)$.

(b) If A is k-alg that is not finitely generated, the closed points need not be dense.

- Look at 3.4K. Consider Spec $k[x]_{(x)}$ where $k[x]_{(x)}$ not finitely generated k-algebra.
- $D(x) = \{(0)\}$ in Spec $k[x]_{(x)}$ as if $x \in \mathfrak{p} \in \operatorname{Spec} k[x]_{(x)}$ then $1 = xx^{-1} \in \mathfrak{p}$. So $D(x)$ does not contain any closed point.

3.6K

- Recall closed points of $\mathbb{A}^n_k$ are maximal ideals of Spec $k[x_1, \ldots, x_n]$, i.e. the classical points of k^n.

  As Spec A is closed in $\mathbb{A}^n_k$ so closed points of Spec A are $k^n \cap \operatorname{Spec} A$.
- Consider f, g different functions on X (i.e. $f, g \in A$), then $f-g$ is nonzero on open set $D(f-g)$.
- $D(f-g)$ is nonempty as $\mathfrak{N}(A) = \{0\}$, otherwise $f-g \in \mathfrak{p}$ for any prime $\mathfrak{p}$, or $f-g \in \mathfrak{N}(A)$, $f-g \neq 0$ contradiction.
- By 3.6J (a), $D(f-g)$ contains a closed point of Spec A $\Rightarrow$ $f-g$ differs at a closed point of Spec A.
- Thus, $f \neq g$ iff exists closed point of X so $f-g$ evaluated at such closed point is nonzero.

3.6.10 Specialisation and generization.

- We show in $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,y]$, $[(y-x^4)]$ is a generization of $[(x-2, y-4)]$. because

  $(y-x^2) \subseteq (x-2, y-4)$ so if $\{[(y-x^4)]\} = V(S)$ then $S \subseteq (y-x^4) \subseteq (x-2, y-4)$ so $(x-2, y-4) \in V(S)$.

$\boxed{36L}$ $\underline{X} = \operatorname{Spec} A$, $[q]$ specialization of $[p]$ then $[q] \in [p] = V(S)$. This follows $S \subseteq p \Rightarrow V(p) \subseteq V(S)$. As $V(S)$ closure of $[p]$ and $p \in V(p)$ so $V(S) \subseteq V(p)$ $\Rightarrow V(S) = V(p) = \overline{[p]} \Rightarrow p \subseteq q$.

$\boxed{36M}$ $[(y-x^2)] \in \mathbb{A}^2$ generic point for $V(y-x^2)$ as $\overline{[(y-x^2)]} = V(y-x^2)$. from 3.6L.

$\boxed{3.6N}$ p generic point for closed subset K.

- For $q \in K$ and U open nbh of q. If $p \notin U$ then $p \in X \setminus U$ closed so $K \subseteq X \setminus U$ so $q \in X \setminus U$, a contradiction. Thus, $p \in U$.
- For $r \notin K = \bigcap_{\substack{p \in U \\ \text{closed}}} U$ so $r \notin U$ for some closed U containing p. $\Rightarrow X \setminus U \ni r$ open and $p \notin X \setminus U$

$\boxed{3.6O}$ Consider partially ordered set $\mathcal{S}$ of irreducible closed subsets of X containing x. Let $\{Z_\alpha\}$ be any totally ordered subset of $\mathcal{S}$. We show $\{Z_\alpha\}$ has an upper bound, i.e. $\bigcup Z_\alpha$ is irreducible (hence its closure is irreducible closed).

- Indeed, suppose $\bigcup Z_\alpha$ not irreducible, then $\exists$ closed proper X_1, X_2 of $\bigcup Z_\alpha$ so $\bigcup Z_\alpha = X_1 \cup X_2$. Because X_1, X_2 proper so $\exists \alpha, \beta$ so $Z_\alpha \not\subseteq X_1$, $Z_\beta \not\subseteq X_2$.

Because α totally ordered so if $\alpha \le \beta$ then $Z_\beta \not\subseteq X_1, X_2$ $\Rightarrow Z_\beta = (Z_\beta \cap X_1) \cup (Z_\beta \cap X_2)$ union of 2 proper closed subsets of $Z_\beta \Rightarrow Z_\beta$ irreducible, contradiction.

- By Zorn's lemma, there exists maximal irreducible closed subset of X containing x.

3.6P $\mathbb{A}^2_{\mathbb{C}}$ is Noetherian topological spaces. is clear from the description of its closed subsets in 3.4.3.

$\mathbb{C}^2$ with classical topology is not Noetherian top space.

$I_n = \{(a,b)\in\mathbb{C} \mid a^2+b^2 \leq \frac{1}{n}\}$ closed in $\mathbb{C}$

and $I_1 \supset I_2 \supset \dots$ never stabilises.

3.6Q Every connected component of topological space X is union of irreducible components of X.

Because every point lie in some irreducible component, it suffices to show if U connected component, V irreducible component and $U \cap V \neq \emptyset$ then $V \subset U$.

- First, we show $U \cup V$ connected. Suppose the contrary $U \cup V = A \cup B$ with A, B disjoint nonempty open of $U \cup V$. $\Rightarrow U = (U \cap A) \cup (U \cap B)$.

As U connected, WLOG, $U \cap A = \emptyset \Rightarrow U \subset B$.

Similarly, for connected V, either $V \cap A$ or $V \cap B$ is $\emptyset$. As A, B nonempty; $A, B \subset U \cup V$ so $V \cap B = \emptyset \Rightarrow V \subset A$.

We have $V \subset A$, $U \subset B$, $A \cap B = \emptyset \Rightarrow U \cap V = \emptyset$ contradiction. Thus $U \cup V$ connected.

- $U \cup V \supset U$ connected so $U \cup V = U$

$\Rightarrow V \subseteq U$, as desired.

- **Claim:** $U \subset X$ both open and closed then U union of connected components.

Proof: With the same strategy, consider V connected component so $V \cap U \neq \phi$. We show $V \subset U$. As U both closed and connected, $V \cap U$ both open and closed in V.

$\Rightarrow V \setminus (V \cap U)$ open and closed in V

$\Rightarrow V = (V \cap U) \cup (V \setminus (V \cap U))$ union of 2 disjoint open sets in V. As V connected, $V \cap U \neq \phi$ so

$V \setminus (V \cap U) = \phi \Rightarrow V \cap U = V \Rightarrow V \subset U$ ┘

- **Claim:** X Noetherian top space, show union of any subset of connected components of X is always open and closed.

Proof: As X is closed, by 3.6.15, X is expressed uniquely as finite union of irreducible closed subsets; none contained in any other, say $X = X_1 \cup \dots \cup X_r$.

- Consider irreducible component Z of X then this follows Z must be one of $X_1, \dots, X_r$. $\Rightarrow$ X has finite # irreducible components.
- Note that 2 connected components must be disjoint (same argument as in 3.6Q) and every connected component is union of irreducible components so if Z is union of subset of irreducible components then so is $X \setminus Z$; and Z and $X \setminus Z$ must be finite union of irreducible components, hence closed $\Rightarrow$ Z both open and closed.

$\boxed{3.6.16}$ Noetherian rings: every ascending seq of ideals eventually stabilises.

$\boxed{3.6R}$ Ring A is Noetherian iff every ideal of A is finitely generated.

- If A Noetherian. Consider ideal I of A. Consider ascending chain of ideals: $I_1 = (a_1)$ for some $a_1 \in A$; for $i > 1$, pick $a_i \in I$ so $a_i \notin I_{i-1}$ and let $I_i = I_{i-1} + (a_i)$

As A Noetherian, this chain must stabilises, i.e. $\exists r$ so $I_r = I_{r+1} = \dots \Rightarrow$ there does not exist $a \in I$ so $a \notin I_{r-1} \Rightarrow I_r = I$ finitely generated.

- If every ideal of A is finitely generated. Consider ascending chain of ideals $I_1 \subset I_2 \subset \dots$ in A.

Then $I = \bigcup I_i$ is an ideal, hence is finitely generated by $a_1, \dots, a_r$. Each a_i is in some I_j so this implies $\exists j$ s.t. $a_1, \dots, a_r \in I_j$

$\Rightarrow I_j = I \Rightarrow I_j = I_{j+1} = \dots = I$.

$\boxed{3.6S}$ If A Noetherian ring then Spec A Noetherian topological space.

Consider descending closed sets of Spec A

$V(S_1) \supset V(S_2) \supset \dots$ where S_i radical ideals of A $\Rightarrow S_1 \subset S_2 \subset \dots$ (we do this by applying $I(\cdot)$, which is introduced in later section).

When $A = k[x_1, \ldots]$ is not Nuetherian, then Spec A is not Nuetherian top space, with descending chain of closed sets $V(x_1) \supset V(x_1, x_2) \supset V(x_1, x_2, x_3) \supset \ldots$

3.6T Every open set of Nuetherian top space is quasicompact.

Let U open set of Noetherian top space X. Say $U = \bigcup_{i \in I} U_i$ union of open U_i of U (hence of X). We form an ascending chain of open sets $V_1 \subset V_2 \subset \ldots$ where each V_j is finite union of U_i. We can do this inductively: If $V_i \neq U$ then $\exists j$ so $U_j \not\subset V_i$, let $V_{i+1} = V_i \cup U_j$.

Hence, $X \setminus V_1 \supset X \setminus V_2 \supset \ldots$ descending chain of closed subsets in X, hence must stabilise. This mean $V_i = V_{i+1} = \ldots$ for some i $\Rightarrow V_i = U$.

$\Rightarrow$ U finite union of open sets.

3.6U If M Noetherian A-module, then every submodule is finitely generated A-module.

Use the same idea as in 3.6R.

3.6V If $0 \to M' \to M \xrightarrow{\phi} M'' \to 0$ exact, show M' and M'' Nuetherian iff M Noetherian.

— If M Nuetherian then $M' \hookrightarrow M$ as A-submodule of M also Noetherian. To show M'' Nuetherian, consider ascending chain of A-submodule of M'':

$I_1 \subset I_2 \subset I_3 \subset \ldots \Rightarrow \phi^{-1}(I_1) \subset \phi^{-1}(I_2) \subset \ldots$ ascending chain of A-submodules of M. $\Rightarrow \ldots$

— If M' and M'' Nuetherian. Consider ascending chain of A-submodules of M: $N_1 \subset N_2 \subset \ldots$

$\Rightarrow$ We have $N_1 \cap M' \subset N_2 \cap M' \subset \ldots$ (1)

$\phi(N_1) \subset \phi(N_2) \subset \ldots$ (2)

This cannot be both equal because of following lemma:

Lem: If $N_1 \cap M' = N_2 \cap M'$ and $\phi(N_1) = \phi(N_2)$, $N_1 \subset N_2$ then $N_1 = N_2$. Indeed, let $x \in N_2$. We show $x \in N_1$. Then as $\phi(N_1) = \phi(N_2)$, exists $y \in N_1$ so $\phi(x) = \phi(y) \Rightarrow x - y \in \ker \phi = M'$

$\Rightarrow x - y \in M' \cap N_2 = M' \cap N_1 \Rightarrow x \in N_1$. ⌋

Because (1) and (2) must stabilise so $N_1 \subset N_2 \subset$ must stabilise..

$\boxed{3.6W}$ Induction : $0 \to A \to A \oplus A \to A \to 0$

$a \mapsto (a,0)$

$(a,b) \mapsto b$

exact so $A^{\oplus 2}$ Noetherian A-mod.

$\boxed{3.6X}$ M is finitely generated A-module.

$\Rightarrow$ $\exists$ quotient map $A^{\oplus n} \xrightarrow{\phi} M$ of A-modules

$\Rightarrow$ exact seq $0 \to \ker\phi \to A^{\oplus n} \to M \to 0$.

As $A^{\oplus n}$ Noetherian A-module, so is M.

§3.7. Function $I(\cdot)$, taking subsets of Spec A to ideals of A

- Given $S \subset \operatorname{Spec} A$, $I(S)$ set of functions vanishing on S.

In other words, $I(S) = \bigcap_{[\mathfrak{p}]\in S} \mathfrak{p} \subset A$. (when S nonempty)

- Observations: $I(S)$ ideal of A, $I(\cdot)$ inclusion reversing

$I(\overline{S}) = I(S)$

3.7A $A = k[x,y]$. If $S = \{[(y)], [(x,y-1)]\}$ then $I(S)$ consists of polynomials vanishing on the y-axis, and at the point $(1,0)$. Find generators for this ideal.

- $I(S)$ has generator yx and $y(y-1)$.

As if $f(x,y) \in (y) \cap (x,y-1)$ then $f(x,y) = y\,g(x,y)$ and $g(x,y) \in (x,y-1)$.

3.7B $S \subseteq \mathbb{A}^3_{\mathbb{C}}$ union of 3 axes. Find generators for S

$$S = \{(x,y), (y,z), (z,x)\} \Rightarrow I(S) = (x,y) \cap (y,z) \cap (z,x)$$

We show $I(S) = (xy, yz, zx)$.

Let $f(x,y,z) \in I(S)$. As $f \in (y,z)$ so $f = yf_1 + zf_2$

As $f \in (x,z)$ so $yf_1 \in (x,z) \Rightarrow f_1 \in (x,z)$

$\Rightarrow f_1 = xf_3 + zf_4 \Rightarrow f = yxf_3 + yzf_4 + zf_2$

As $f \in (x,y)$ so $zf_2 \in (x,y) \Rightarrow f_2 \in (x,y)$

$\Rightarrow f_2 = xf_5 + yf_6$.

Thus, $f \in (xy, yz, zx)$. We are done

3.7C $\cdot\, V(I(S)) = \{[q] \in \operatorname{Spec} A \mid \bigcap_{p \in S} p \subset q\}$ contains S so $V(I(S)) \supset \overline{S}$.

- Let $V(T)$ be closed set containing S then

$$T \subset \bigcap_{p \in S} p \Rightarrow T \subset q \text{ for } q \in V(I(S))$$

$$\Rightarrow V(I(S)) \subset V(T) \Rightarrow V(I(S)) \subset \overline{S}.$$

3.7D Use 3.4J.

3.7E

$$\begin{tikzcd} \text{Irreducible closed of Spec}A \arrow[r, shift right, "I"'] & \text{prime ideals of} A \arrow[l, shift right, "V"'] \end{tikzcd}$$

- for p prime ideal of A then $V(p) = \overline{[p]}$ by 3.6.I. This is irreducible because if $\overline{[p]} = V(S) \cup V(T)$ then one of closed subsets $V(S), V(T)$ must contain $[p]$ $\Rightarrow V(S) \supset \overline{[p]} \Longrightarrow V(S) = \overline{[p]}$.
- Contrapositively, if $I(S)$ is not prime ideal of A for some irreducible closed S of Spec$A$.

Then exists $a, b \notin I(S)$ but $ab \in I(S)$.

$$\Rightarrow I(S) = ((a) + I(S)) \cdot ((b) + I(S)).$$

$$\Rightarrow S = V(I(S)) = V((a) + I(S)) \cup V((b) + I(S)).$$

As S irreducible, $V((a) + I(S)) = S$

$$\Rightarrow (a) + I(S) \subset \sqrt{(a) + I(S)} = I(S).$$

$$\Rightarrow a \in T(S).$$

- Finally, $V(I(S)) = S$ and $I(V(p)) = I(\overline{[p]}) = \bigcap_{q \subset \overline{[p]}} q = p$. for prime ideal p and irreducible closed S of A.

3.7F

$$\begin{tikzcd} \text{Irreducible components of Spec}A \arrow[r, shift right, "I"'] & \text{Minimal prime ideal of} A \arrow[l, shift right, "V"'] \end{tikzcd}$$

True by 3.7E and inclusion reversing of $V(\cdot), I(\cdot)$.

3.7G Minimal prime of $k[x,y]/(xy)$ corresponding to minimal prime of $k[x,y]$ containing (xy).

Recall primes of $k[x,y]$ are either (0), $(x-a, y-b)$ or $(f(x,y))$ for $f(x,y)$ irreducible.

$\Rightarrow (x), (y)$ are the minimal prime of $k[x,y]/(xy)$.

Chapter 4 : The structure sheaf, and the def of schemes

4.1 Structure of affine scheme.

* **Def:** $\mathcal{O}_{\operatorname{Spec} A}(D(f))$ be localisation of A at multiplicative set of all functions that do not vanish outside of $V(f)$.
(i.e. functions $g \in A$ whose vanishing points are vanishing points of f).

4.1A Show isomorphism $A_f \longrightarrow \mathcal{O}_{\operatorname{Spec} A}(D(f))$

It suffices to show this map is surjective.

Consider $g/h \in \mathcal{O}_{\operatorname{Spec} A}(D(f))$ with $V(h) \subset V(f)$ or $D(f) \subset D(h)$. By 3.5E then $f^n \in (h)$ or $hk = f^n$ for $k \in A$. $\Longrightarrow g/h = gk/f^n \in A_f$.

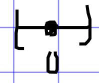
4.1B Replace A by A_f and note $\operatorname{Spec} A_f = D(f)$.

§4.2. Visualising schemes II: nilpotents.

— How to visualise $\operatorname{Spec} \mathbb{C}[x]/(x^2)$? As subset of $\operatorname{Spec} \mathbb{C}[x] = \mathbb{A}^1_{\mathbb{C}}$, this is just the origin $0 = [(x)]$.

— From exercise 3.2T, image of polynomial $f(x)$ is the information its value at 0, and its derivative (consider $f(x)$ in $k[x,\epsilon]/(\epsilon^2)$ then $f(x+\epsilon) = f(x) + \epsilon f'(x)$).

$\Longrightarrow$ Vakil suggested to picture this as



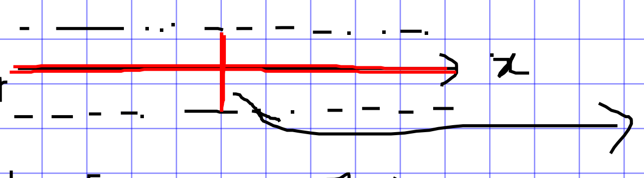
— See picture 4.3.

- As $\operatorname{Spec} \mathbb{C}[x,y]/(y^2, xy) = \operatorname{Spec} \mathbb{C}[x,y]/(y^2) \cap \operatorname{Spec} \mathbb{C}[x,y]/(xy)$

$\{(y), (y, x-a)\}$ $\quad (y), (y, x-a) \quad (x), (y), (x,y)$

$(x-a, y), (x, y-a)$.

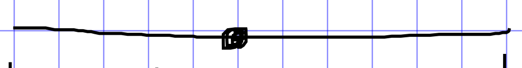
so picture of $\operatorname{Spec} \mathbb{C}[x,y]/(y^2, xy)$ is



all the points of $\operatorname{Spec} \mathbb{C}[x,y]/(y^2, xy)$.

A fuzz at the origin as we have y^2 which thickens the x-axis, then we have $xy = 0$ and the two intersect as such.

But Vakil indicates this as a circle around origin?



In other words, knowing what polynomial in $\mathbb{C}[x,y]$ is modulo (y^2, xy) is the same as knowing its value on x-axis and first order differential info about the origin.

NEED TO CHECK THIS:

But how to see this right away: $(y^2, xy) = (y) \cdot (x, y)$

x-axis $\quad$ fuzz at origin

§4.3. Definition of schemes.

43A — We start from isomorphism $\pi : \overbrace{\mathrm{Spec}A}^{X} \to \overbrace{\mathrm{Spec}A'}^{X'}$ of affine schemes then this induces isomorphism $\mathcal{O}_X \to \pi^*\mathcal{O}_X$ of sheaves of ring over X'.

Taking global section gives morphism (not necessarily isomorphism because surjectivity is not preserved when taking sections) $A' \to A$.

But we also have inverse map $\pi^*\mathcal{O}_X \to \mathcal{O}_{X'}$, and by taking global section of this, we have $A \to A'$, which should be inverse to previous map.

— We start from isomorphism of rings $A' \xrightarrow{\pi} A$. We construct $\mathrm{Spec}A \xrightarrow{\phi} \mathrm{Spec}A'$ as follows:

$\phi(\mathfrak{p}) = \pi^{-1}(\mathfrak{p})$. Because π bijective, this map also bijective.

⊕ Next, we show ϕ is continuous.

For $f \in \mathrm{Spec}A'$ then

$$\phi^{-1}(D(f)) = \{\mathfrak{p} \in \mathrm{Spec}A \mid \phi(\mathfrak{p}) = \pi^{-1}(\mathfrak{p}) \not\ni f\}$$
$$= \{\mathfrak{p} \in \mathrm{Spec}A \mid \mathfrak{p} \not\ni \pi(f)\}$$
$$= D(\pi(f)) \subseteq \mathrm{Spec}A.$$

⊕ Next, we show ϕ induces an isomorphism of sheaves $\mathcal{O}_X$ and $\mathcal{O}_{X'}$ where $X = \mathrm{Spec}A$, $X' = \mathrm{Spec}A'$. Indeed, we define map

$$\mathcal{O}_{X'}(D(f)) \longrightarrow \phi^*\mathcal{O}_X(D(f)) = \mathcal{O}_X(D(\pi(f)))$$
$$A'_f \longrightarrow A_{\pi(f)} \quad \text{via } A' \xrightarrow{\pi} A$$

which is compatible with restrictions, hence inducing $\mathcal{O}_{X'} \longrightarrow \mathcal{O}_X$.

⊕ By considering $\pi^{-1}: A \to A'$, we can repeat above process to define ϕ^{-1} showing it is an isomorphism of affine schemes.

- To check this gives bijection between isomorphism $A' \to A$ and $\mathrm{Spec}A \to \mathrm{Spec}A'$, it suffices to show if we start with $A' \xrightarrow{\pi} A$ and get $\mathrm{Spec}A \xrightarrow{\phi} \mathrm{Spec}A'$, and from this get $A' \xrightarrow{\rho} A$ from ϕ then $\pi = \rho$.

But this is clear from our construction of $\mathrm{Spec}A \to \mathrm{Spec}A'$ once we have $A' \to A$.

4.3B Natural isomorphism of ringed spaces
$$\left(D(f), \mathcal{O}_{\operatorname{Spec} A}|_{D(f)}\right) \cong \left(\operatorname{Spec} A_f, \mathcal{O}_{\operatorname{Spec} A_f}\right).$$

— We have natural isomorphism $\Gamma(D(f), \mathcal{O}_{\operatorname{Spec} A}) \cong A_f$, which induces isomorphism above, from previous exercise

— Moreover, this isomorphism is natural as this comes from the inclusion $\operatorname{Spec} A_f \longrightarrow \operatorname{Spec} A$ of sets, where Zariski top of $\operatorname{Spec} A$ restrict to that of $\operatorname{Spec} A_f$ (3.4.I) and the structure sheaf of $\operatorname{Spec} A$ restricts to the structure sheaf of $\operatorname{Spec} A_f$.

4.3C X scheme, U open then $(U, \mathcal{O}_X|_U)$ is also a scheme.

— We know $(U, \mathcal{O}_X|_U)$ is a sheaf of rings. It suffices to show $(U, \mathcal{O}_X|_U)$ is locally isomorphic to affine scheme

— Pick $x \in U$ then exists open nbh V of X containing x so $(V, \mathcal{O}_X|_V)$ is affine scheme.

Then $V \cap U$ is open of V containing x, which will be covered by distinguished basis $D(f)$ of V. One of these $D(f)$ must contain x.

$\Longrightarrow$ $\left(D(f), \mathcal{O}_X|_{D(f)}\right)$ is affine from 4.3B and $D(f) \subset V \cap U \subset U$.

4.3D This follows from proof of 4.3. For every open U, $(U, \mathcal{O}_X|_U)$ is a scheme, hence we can cover it by affine open subset of U, hence of X.

$\boxed{4.3E}$ X, Y scheme then $X \sqcup Y$ has open sets $U \sqcup Y$ and $X \sqcup V$ for U, V open in X, Y respectively. The structured sheaf of $X \sqcup Y$ is

$$\mathcal{O}_{X\sqcup Y}((U\sqcup Y)\cup(X\sqcup Y)) = \mathcal{O}_X(U) \cup \mathcal{O}_Y(V).$$

(a) We have homeomorphism

$$X = \operatorname{Spec} A_1 \sqcup \operatorname{Spec} A_2 \xrightarrow{\pi} \operatorname{Spec} A_1 \times A_2 = Y$$

$$\mathfrak{p}_1 \longmapsto \mathfrak{p}_1 \times A_2$$

$$\mathfrak{p}_2 \longmapsto A_1 \times \mathfrak{p}_2.$$

We need to define isomorphism $\mathcal{O}_Y \longrightarrow \pi^*\mathcal{O}_X$ of sheaves over Y. Note Y has distinguished basis $D(f) \times A_2$ for $f \in A_1$ and $A_1 \times D(f)$ for $f \in A_2$. WLOG, $f \in A_1$.

$$\mathcal{O}_Y(D(f)\times A_2) \longrightarrow \pi^*\mathcal{O}_X(D(f)\times A_2) = \mathcal{O}_X(D(f))$$

$$(A_1)_f \times A_2 = (A_1 \times A_2)_f \longmapsto (A_1)_f$$

and so we can define this map naturally to be the projections.

$\boxed{4.3G}$ (a) $f \in \mathcal{O}_X(X)$. We need to show $\{p \in X \mid f_p \in m_p\}$ is closed. Note $f_p \in m_p$ is equivalent to f_p not invertible in $\mathcal{O}_{X,p}$ as $\mathcal{O}_{X,p}$ is a local ring. Thus, it suffices to show $\{p \in X \mid f_p \text{ invertible in } \mathcal{O}_{X,p}\}$ is open. This actually holds for any ringed space X.

Let $p \in X$ so $f_p \in \mathcal{O}_{X,p}$ invertible, then there exists $s_p \in \mathcal{O}_{X,p}$ corresponding to $(s \in \mathcal{O}_X(U), p \in U)$. Such that $s_p f_p = 1 \Rightarrow s \cdot f|_U = 1$ on $\mathcal{O}_X(U)$ (by def of stalk) $\Rightarrow f_q$ is invertible on $\mathcal{O}_{X,q}$ for all $q \in U$, as desired.

(b) Suppose $f_p \notin m_p$, i.e. f_p is invertible on $\mathcal{O}_{X,p}$ for all $p \in X$. We need to show f is invertible.

With the same argument, for each p, we can find $s_p = (s \in \mathcal{O}_X(U_p), p \in U_p)$ so $fs = 1$ on U_p.
$\Rightarrow$ glue these s_p together, we obtain inverse for f.

– For point $[(p)]$ in Spec A, the residue field at p is A_p/pA_p, which is isomorphic to $K(A/p)$, the fraction field of the quotient.

$\Longrightarrow$ The idea of "function vanished at a point" in Spec A agrees with our original point of view about this for Spec A (i.e. just take mod p instead of localising).

$\boxed{4.3.7}$ $\mathcal{F}$ an $\mathcal{O}_X$-module of scheme X (or any locally ringed space), the fibre of $\mathcal{F}$ at $p \in X$ is

$$\mathcal{F}|_p := \mathcal{F}_p \otimes_{\mathcal{O}_{X,p}} k(p).$$

Here $\mathcal{F}_p$ is $\mathcal{O}_{X,p}$-module, and $\mathcal{O}_{X,p} \twoheadrightarrow k(p)$ so $\mathcal{F}|_p$ as $k(p)$-module.

§4.4. Three examples

$\boxed{4.4.1}$ The plane minus the origin.

– $U = \mathbb{A}^2_k - \{[(x,y)]\}$ is open of $\mathbb{A}^2_k$, hence a scheme.

– Functions on U are $k[x,y]$., but U is not affine.

– How to find $\Gamma(U, \mathcal{O}_{\mathbb{A}^2})$: Note $U = D(x) \cup D(y)$.

By def of sheaves, element $f \in \Gamma(U, \mathcal{O}_{\mathbb{A}^2})$ corresponds exactly to a pair of functions f_x, f_y over $D(x)$ and $D(y)$ so that they agree on $D(xy) = D(x) \cap D(y)$.

As in Vakil's notes, such pair are (p, p) where $p \in k[x,y]$

$\Longrightarrow \Gamma(U, \mathcal{O}_{\mathbb{A}^2}) \equiv k[x,y]$.

$\boxed{4.4.4}$ Gluing 2 copies of $\mathbb{A}^1$ in 2 different ways.

$\boxed{4.4A}$ Given:
- schemes X_i
- open subschemes $X_{ij} \subset X_i$ with $X_i = X_{ii}$
- iso $f_{ij} : X_{ij} \to X_{ji}$ with f_{ii} identity

such that: (cocycle condition) isomorphism agree on triple intersections, i.e. $f_{ik}|_{X_{ij} \cap X_{ik}} = f_{jk}|_{X_{ji} \cap X_{jk}} \circ f_{ij}|_{X_{ij} \cap X_{ik}}$.

(This condition ensures that f_{ij} and f_{ji} are inverses, so that we can glue X_{ij} in X_i with X_{ji} in X_j).

Show that there is a unique scheme X (up to unique iso) along open subsets isomorphic to X_i :

- As a set $X = \bigsqcup_i X_i \Big/ \left(X_{ij} \overset{f_{ij}}{\simeq} X_{ji} \,;\, X_{ij} \cap X_{ik} \sim X_{jk} \cap X_{ji} \sim X_{ki} \cap X_{kj} \right)$ with the quotient topology.
- Glue sheaves $(X_i, \mathcal{O}_{X_i})$ together, we get X.

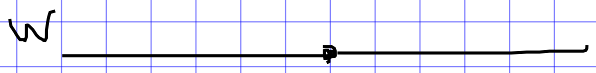
$\boxed{4.4B}$ Show affine line with double origin is not affine.

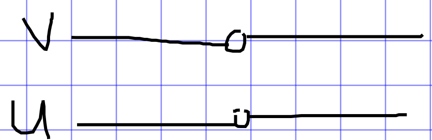
- Recall how to define affine line with double origin :

$X = \operatorname{Spec} k[t]$, $Y = \operatorname{Spec} k[u]$, $U = D(t) = \operatorname{Spec} k[t, \frac{1}{t}]$

$V = \operatorname{Spec} k[u, \frac{1}{u}]$.

$\underbrace{V \simeq U \text{ via } t \leftrightarrow u}_{\text{this will give us isomorphism of ringed spaces}}$. and we glue X and Y along $V \simeq U$





Call resulting scheme W.

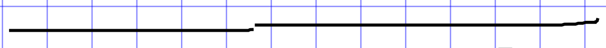
- We compute global section of W: This corresponds to a pair of func f on X and g on Y so that they agree on $U \sim V$.

i.e. $f \in k[t]$, $g \in k[u]$ such that $f \in k[t, \frac{1}{t}]$ is $g \in k[u, \frac{1}{u}]$ via $t \sim u$. $\Longrightarrow f, g$ same pol

$\Rightarrow \Gamma(W, \mathcal{O}_W) \equiv k[x]$. (where $\Gamma(W, \mathcal{O}_W) \longrightarrow \Gamma(X, \mathcal{O}_W)$, $x \longmapsto t$)

- If W is affine then $W \simeq \mathbb{A}^1_k$.

In $\mathbb{A}^1_k$, prime ideal (x) corresponds to a point in $\mathbb{A}^1_k$ by taking vanishing set V of (x).

But in W,  2 points in W

as $V(x) = \{ a \in W \,/\, x_a = 0 \} = \{ (t), (u) \}$.

$\boxed{4.4C}$

$A_1 = \operatorname{Spec} k[x_1, y_1]$, $A_2 = \operatorname{Spec} k[x_2, y_2]$
and $U_1 = D(x_1) \cup D(y_1)$, $U_2 = D(x_2) \cup D(y_2)$

We glue A_1 and A_2 alang $D(x_1) \sim D(x_2)$ (via $x_1 \longleftrightarrow x_2$, $y_1 \longleftrightarrow y_2$)
$D(y_1) \sim D(y_2)$

(Imagine $D(x_1)$ as affine "plane" minus y_1-axis then $D(x_1) \cup D(y_1)$ should be everything except a point (x_1, y_1)).

Need to check that this glues give us isomorphism $(U_1, \mathcal{O}_{A_1}) \cong f_*(U_2, \mathcal{O}_{A_2})$ i.e. if $D(x_1) \xleftrightarrow{f_x} D(x_2)$, $D(y_1) \xleftrightarrow{f_y} D(y_2)$

then need to check $f_x|_{D(x_1) \cap D(y_1)} = f_y|_{D(x_1) \cap D(y_1)}$

$= D(x_1 y_1)$ $\qquad = D(x_2 y_2)$

$\boxed{4.4.6}$ Projective line.

Glue $X = \operatorname{Spec} k[t]$ with $Y = \operatorname{Spec} k[u]$ along
$U = \operatorname{Spec} k[t, \frac{1}{t}] \sim V = \operatorname{Spec} k[u, \frac{1}{u}]$ via $u \leftrightarrow \frac{1}{t}$

- Imagine this when k is alg closed:

$(t-a) \in U$ is glued to $[(\frac{1}{u} - a)] = [(u - \frac{1}{a})] \in V$.

$(0) \in U$ ———— $(0) \in V$

$\boxed{4.4D}$

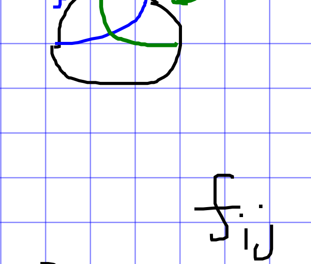
$\mathbb{P}^n_k$ by glueing $n+1$ open sets U_i, each isomorphic to $\mathbb{A}^n_k$. $U_i = \operatorname{Spec} k[x_{0/i}, x_{1/i}, \dots, x_{n/i}]/(x_{i/i} - 1)$.

- Glue $D(x_{j/i})$ of U_i to $D(x_{i/j})$ of U_j via:

$$\operatorname{Spec} k[x_{0/i}, x_{1/i}, \dots, x_{n/i}, 1/x_{j/i}]/(x_{i/i} - 1) \cong$$
$$\operatorname{Spec} k[x_{0/j}, \dots, x_{n/j}, 1/x_{i/j}]/(x_{j/j} - 1)$$

by $x_{k/i} \xmapsto{f_{ij}} x_{k/j}/x_{j/i}$ and $x_{k/j} \xrightarrow[f_{ji}]{} x_{k/i}/x_{i/j}$.

- Check glueing agrees over triple intersection, i.e.



$$D(x_{j/i}) \cap D(x_{k/i}) = D(x_{j/i} x_{k/i}).$$

$$f_{jk}|_{D(x_{j/i} x_{k/i})} \circ f_{ij}|_{D(x_{j/i} x_{k/i})}$$

$$= f_{ik}|_{D(x_{j/i} x_{k/i})}.$$

$$x_{h/i} \xmapsto{f_{ij}} x_{h/j}/x_{i/j} \xmapsto{f_{jk}} (x_{h/k}/x_{j/k})/(x_{i/k}/x_{j/k})$$
$$= x_{h/k}/x_{i/k}.$$

4.4E Element in $\Gamma(\mathbb{P}^n_A, \mathcal{O})$ corresponds to section over U_i that agrees on overlap.

Let $f \in \Gamma(U_0, \mathcal{O})$ and $g \in \Gamma(U_1, \mathcal{O})$. Restricting to overlap is still f and g, and they must agree via $f\left({x_{0/1}}/{x_{0/1}}, {x_{1/1}}/{x_{0/1}}, \dots, {x_{n/1}}/{x_{0/1}} \right)$

$= g(x_{0/1}, \overset{=1}{x_{1/1}}, \dots, x_{n/1})$. $\Longrightarrow$ happens only when $f = g = \text{const}$.

4.4F When $a_i \neq 0$, $[a_0, \dots, a_n]$ identified with point $(a_0/a_i, \dots, a_n/a_i)$ in U_i, which makes sense when considering overlap: $a_i, a_j \neq 0$ then $(a_0/a_i, \dots, a_n/a_i)$ in U_i same as $(a_0/a_j, \dots, a_n/a_j)$ in U_j.

4.4.11 $\operatorname{Spec} \mathbb{Z}/60\mathbb{Z} = \{(2), (3), (5)\}$

- These are closed points as they are maximal ideals of $\mathbb{Z}/60\mathbb{Z}$ $\Rightarrow$ $\operatorname{Spec}\mathbb{Z}/60\mathbb{Z}$ has discrete top.
- Stalks are $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/5\mathbb{Z}$.

Indeed, stalk at (2) is $\mathbb{Z}/60\mathbb{Z}$ localised at (2). To be explicit, element is of the form $\frac{a}{b}$ where $a, b \in \mathbb{Z}/60\mathbb{Z}$. $2 \nmid b$.
And $\frac{a}{b} = \frac{c}{d}$ when $(ad - bc)e \underset{\text{mod } 60}{=} 0$ for some $2 \nmid e$.
This is equivalent to $ad - bc \equiv 0 \bmod 4$ or $\frac{a}{b} \equiv \frac{c}{d} \bmod 4$ as $2 \nmid bd$. $\Rightarrow$ Stalk is really $\mathbb{Z}/4\mathbb{Z}$.

- Global section of $\mathbb{Z}/6\mathbb{Z}$ corresponds to a section at (2), (3) and (5) (no compatibility as these are discrete topology)

$\Rightarrow \mathbb{Z}/60\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

§4.5. Projective schemes, and the Proj construction

4.5A Recall from 4.4F, $\mathbb{P}^2_k$ can be identified with points $[x_0 : x_1 : x_2]$ etc. Hence, we now make sense of "scheme cut out by $x_0^2 + x_1^2 - x_2^2 = 0$ in $\mathbb{P}^2_k$":

- If $[x_0 : x_1 : x_2] \in U_2$, i.e. $x_2 \neq 0$ that satisfies $x_0^2 + x_1^2 - x_2^2 = 0$ then viewing $U_2 = \operatorname{Spec} k[x_{0/2}, x_{1/2}]$, these points must satisfy $x_{0/2}^2 + x_{1/2}^2 - 1 = 0$.

$\Rightarrow$ suggests $V_2 = \operatorname{Spec} k[x_{0/2}, x_{1/2}]/(x_{0/2}^2 + x_{1/2}^2 - 1)$ should be the affine open of our scheme.

- Similarly for U_0, U_1 to get V_0, V_1. And the glueing between V_i should be the same as for U_i. (see 4.4D).

4.5B Similar for 4.5A.

4.5C (a) Ideal I homogeneous $\Leftrightarrow$ if $x = \sum x_n \in I$ then $x_n \in I$ $\forall n$.

"$\Leftarrow$" is clear. For "$\Rightarrow$", suppose I homogeneous, and let $x = \sum x_n \in I$. Because I generated by homogeneous elements so $x = \sum a_n$ where $a_n \in I \cap S_n$.

$\Rightarrow x_n = a_n \in I$, as desired.

- This follows homogeneous ideal $I = \bigoplus_n I_n$.
- $S_\bullet/I$ has natural $\mathbb{Z}$-grading.

$$S_\bullet/I = \bigoplus_n S_n/I_n$$

(b) Use (a).

(c) If homogeneous $I \neq S_\bullet$ s.t. for any homogeneous element $a, b \in S_\bullet$, if $ab \in I$ then $a \in I$ or $b \in I$.

<u>Proof</u>: Consider $b = b_{i_1} + \dots + b_{i_k}$, $a = a_{j_1} + \dots + a_{j_h}$ with $ab \in I$. WLOG $i_1 < i_2 < \dots < i_k$ and $j_1 < \dots < j_h$. By (a), we must $b_{i_k} a_{j_h} \in I \Rightarrow$ either $b_{i_k} \in I$ or $a_{j_h} \in I$. WLOG, $b_{i_k} \in I$ then we follow $(b - b_{i_k})a \in I$. Keep doing the same thing for this. Repeat until find either $b_{i_x} \in I \ \forall x$ or $a_{j_x} \in I$ for all x.

- If T multiplicative subset of $S_\bullet$ containing only homogeneous elements then $T^{-1}S$ has natural structure of $\mathbb{Z}$-graded ring.

Let T_n = elements in T of deg n.

Then $$T^{-1}S = \bigoplus_n \left\{ \frac{s_n}{t_n} : s_n \in S_n, t_n \in T_n \right\}$$

4.5D (a) If $S_\bullet$ is finitely generated graded ring over A then S_+ is finitely generated ideal, generated by $f_1, \ldots, f_n \Rightarrow S_\bullet = A \oplus S_+$ is finitely generated A-algebra

The converse is similar.

(b) If $S_\bullet$ Nuetherian ring then ideal S_+ is finitely generated $\Rightarrow S_\bullet$ finitely generated graded ring.

Further, to show A Nuetherian. Consider ascending chain $I_1 \subset I_2 \subset \cdots$ of ideals of A then $I_1 \oplus S_+ \subset I_2 \oplus S_+ \subset$ ascending chain of ideals of $S_\bullet$ so must stabilses .....

Conversely, if A Nuetherian and S_+ finitely generated ideal. Note I ideal of $S_\bullet$ then $I = U \oplus V$ where $U = I \cap A$ ideal of A and $V = S_+ \cap I$ ideal of S_+. So we are done.

4.5E (a) A is graded $\mathbb{Z}$-ring with invertible element f of positive degree. There is bijection

$$\left\{\begin{matrix}\text{prime ideals} \\ \text{of } A_0\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{homogeneous prime} \\ \text{ideal of } A\end{matrix}\right\}$$

- Use $A_0 \to A$ to associate each homogeneous prime ideal P of A with $P_0 = A_0 \cap P$ prime ideal of A_0.
- For the converse, consider prime ideal P_0 of A_0. We construct homogeneous prime ideal P of A so $P \cap A_0 = P_0$. $P = \bigoplus Q_i$ with $Q_i \subset A_i$ where $a_i \in Q_i$ iff $a_i^{\deg f}/f^i$ is in P_0.
  - P is an ideal because P_0 is an ideal of A_0.
  - P is homogeneous ideal because it is generated by homogeneous elements
  - P is prime: Let $ab \in P$. Because $P = \bigoplus Q_i$ and if $a = a_{i_1} + \ldots + a_{i_k}$, $b = b_{j_1} + \ldots + b_{j_l}$ $\quad i_1 < \cdots < i_k$, $j_1 < \cdots < j_l$ then $a_{i_k} b_{j_l} \in Q_{i_k + j_l}$. As P_0 is prime, either $a_{i_k} \in P$ or $b_{j_l} \in P$. Repeat this process for $(a - a_{i_k}) b \in P$ or $a(b - b_{i_k}) \in P$.

(b) $(S_\bullet)_f$ is $\mathbb{Z}$-graded ring with invertible elements f of positive deg $\Rightarrow$ bijection between primes of $((S_\bullet)_f)_0$ and homogeneous prime ideals of $(S_\bullet)_f$.

$\Rightarrow$ Can view prime ideals of $((S_\bullet)_f)_0$ as homogeneous prime ideals of $(S_\bullet)_f$, which corresponds to homogeneous prime ideals of $S_\bullet$ not containg $f \in S_+$, which is a subset of $\operatorname{Proj} S_\bullet$.

$\Rightarrow$ How to picture $\operatorname{Proj} S_\bullet$?

4.5F T set of homogeneous elements of $S_\bullet$ of positive degree, the (projective) vanishing set of T, $V(T) \subset \operatorname{Proj} S_\bullet$ be homogeneous prime ideals containing T but not S_+

$D(f) := \operatorname{Proj} S_\bullet \setminus V(f)$ projective distinguished set for f homogeneous of pos deg.

$$\begin{tikzcd} D(f) = \left\{\begin{array}{l}\text{homogeneous prime}\\ \text{ideal } P \text{ of } S_\bullet \text{ s.t.}\\ f \notin P \Rightarrow P \not\supset S_+\end{array}\right\} \arrow[r, leftrightarrow] & \left\{\begin{array}{l}\text{homogeneous prime}\\ \text{ideal of } (S_\bullet)_f\end{array}\right\} \arrow[dl, leftrightarrow, "4.5F"] \\ \left\{\begin{array}{l}\text{prime ideal of}\\ ((S_\bullet)_f)_0\end{array}\right\} & \end{tikzcd}$$

4.5G Any open set of $\operatorname{Proj} S_\bullet$ is complement of $V(U)$ for $U \subseteq S_\bullet$, which is also union of $\bigcup_{f \in U} D(f)$.

4.5H (a) If $f^n \in I$ for some n then $f \in \mathfrak{p}$ for any prime containing I $\Rightarrow$ f vanishes on $V(I)$

To really use 3.4J, which traces back to 3.4F and 3.2.12, we need also to mimic these results.

- Mimic 3.2.12. For graded ring $S_\bullet$, the nilradical of $S_\bullet$ is intersection of all homogeneous prime ideals

 Proof: From 3.2.12, we know $\mathfrak{N}(S_\bullet) \subset \bigcap_{P \subset S_\bullet} P$ over hom. P. For the converse, consider $x \notin \mathfrak{N}(S_\bullet)$ we show there exists homogeneous prime P not containing x.
 Consider $(S_\bullet)_x$ whose ______________ corresponds to prime ideal of $((S_\bullet)_x)_0$ by 4.5(e). Because $((S_\bullet)_x)_0$ is not zero-ring, there exists prime ideal in $((S_\bullet)_x)_0$, as desired ∎

- Mimic 3.4F: for I homogeneous ideal of $S_\bullet$, the nilradical $\sqrt{I}$ of I is intersection of all homogeneous prime containing I.

 Proof: Consider graded ring $S_\bullet/I$ and apply previous result.
 Note homogeneous prime of $S_\bullet/I$ corresponds to homogeneous prime of $S_\bullet$ containing I. ┘

- Back to the problem: If f vanishes at $V(I) = \operatorname{Proj} S_\bullet/I$ then f lies in any homogeneous prime of $S_\bullet/I$, i.e. f lies in the nilradical of S/I, which is $\sqrt{I}$. We are done.

(b) $Z \subset \operatorname{Proj} S_\bullet$. $I(Z) = \{f \in S_\bullet \mid f \in \mathfrak{p} \text{ for all } \mathfrak{p} \in Z\}$
$I(Z)$ homogeneous ideal of $S_\bullet$ because all $\mathfrak{p} \in Z$ are homogeneous ideal$_\mathfrak{p}$.

(c) $Z \subset \operatorname{Proj} S_\bullet$. We show any closed subset contains $V(T)$ then it must also contain $V(I(Z))$, implying $V(I(Z)) = \overline{Z}$.
Indeed, if $V(T) \supset Z$ then it means for all $\mathfrak{p} \in Z$, $T \subset \mathfrak{p}$. This follows $T \subset I(Z) \Rightarrow V(T) \supset V(I(Z))$, as desired.

$\boxed{4.5\text{I}}$

- (a) $\Leftrightarrow$ (b) : If $\bigcup D(f_i) \subsetneq \operatorname{Proj} S_\bullet$ iff exists homogeneous prime $\mathfrak{p}$ s.t. $\mathfrak{p} \notin \bigcup D(f_i) \Leftrightarrow f_i \in \mathfrak{p}\ \forall i \Leftrightarrow I \subset \mathfrak{p}$. $\Leftrightarrow V(I) \neq \phi$.
- Note, by proof of 4.5H, $\sqrt{I}$ = intersection of all homogeneous prime containing I, where S_+ is one of them, so (a) $\Leftrightarrow$ (c) is clear.

$\boxed{4.5\text{J}}$ We show closed subset $V(I_0)$ of $\operatorname{Spec}((S_\bullet)_f)_0$ corresponds to $V(I) \cap D(f)$ where $V(I)$ closed of $\operatorname{Proj} S_\bullet$.

$$V(I_0) \text{ of } \operatorname{Spec}((S_\bullet)_f)_0 = \left\{ \begin{array}{l} \text{prime ideal } P_0 \text{ of } ((S_\bullet)_f)_0 \\ \text{s.t. ideal } I_0 \subset P_0 \end{array} \right\}$$

$$\xrightarrow{4.5E} \left\{ \begin{array}{l} \text{homogeneous prime } P = \bigoplus P_i \\ \text{of } (S_\bullet)_f \text{ s.t. homogeneous ideal} \\ I = \bigoplus I_n \text{ obtained via } I_0 \text{ s.t.} \\ I \subset P \end{array} \right\}$$

note to recover $I = \bigoplus I_n$ we only need to know $I \cap S_+$, as if $a \in I$ then $af^k \in I \cap S_+$ for large enough k.

$$\rightsquigarrow \left\{ \begin{array}{l} \text{homogeneous prime } P \cap S_\bullet \text{ of } S_\bullet \\ \text{s.t. } \underline{I \cap S_+ \subset S_+} \text{ homogeneous ideal} \\ I \cap S_+ \subset P \cap S_\bullet \text{ and } f \notin P \cap S_\bullet \end{array} \right\}$$

$$= V(I \cap S_+) \cap D(f).$$

$\boxed{4.5\text{K}}$ We have isomorphism of structure sheaf between $\operatorname{Spec}((S_\bullet)_{fg})_0$ and open $D(g^{\deg f}/f^{\deg g})$ of $\operatorname{Spec}((S_\bullet)_f)_0$.

This is obtained via isomorphism

$$((S_\bullet)_{fg})_0 \xrightarrow{\sim} (((S_\bullet)_f)_0)_{g^{\deg f}/f^{\deg g}}.$$

$$\begin{array}{c} \deg h = \\ k(\deg f + \deg g) \end{array} \quad \frac{h}{(fg)^k} \longmapsto \left(\frac{h g^{m \deg f - k}}{f^{m \deg g + k}} \right) \Big/ \left(g^{\deg f} / f^{\deg g} \right)^m$$

for $m \in \mathbb{Z}_{>0}$ s.t. $m \deg f > k$.

4.5L Showing compatible on triple overlap.

$$\begin{tikzcd}
\Phi_{fg} : \operatorname{Spec}((S_\bullet)_f)_0 \big|_{D(g^{\deg f}/f^{\deg g})} \arrow[r, "\sim"] & \operatorname{Spec}((S_\bullet)_{fg})_0 \arrow[dl, "\wr"] \\
\operatorname{Spec}((S_\bullet)_g)_0 \big|_{D(f^{\deg g}/g^{\deg f})} &
\end{tikzcd}$$

4.5M OK.

4.5N $D(x_i)$ cover $\operatorname{Proj} A[x_0, \ldots, x_n]$ because from 4.5I.

4.5O $\mathbb{P}^n_k$ in §4.4.9 isomorphic to $\operatorname{Proj} k[x_1, \ldots, x_n]$.

Each point of $[a_0, \ldots, a_n]$ of classical projective space corresponds to homogeneous prime ideal $(a_i x_j - a_j x_i)$

Because if $[a_0, \ldots, a_n] \in D(x_0) = \operatorname{Spec} k[x_{i/0}]$, i.e. corresponds to $(x_{i/0} - a_i)$ then via inclusion $D(x_0) \hookrightarrow \operatorname{Proj} k[x_0, \ldots x_n]$, this corresponds to $(a_i x_j - a_j x_i)$.

4.5P I don't think I used the fact $S_\bullet$ generated in deg 1 anywhere (?)

For I homogeneous ideal of S_+, $V(I)$ has scheme structure of $\operatorname{Proj} S_\bullet/I$.

Homogeneous prime $\mathfrak{p}$ in $S_\bullet$ containing I but not S_+ will correspond to homogeneous prime in $S_\bullet/I$ not containing S_+/I

4.5.9 Projective scheme over A = $\operatorname{Proj} S_\bullet$ for finitely generate graded ring over A.

Quasiprojective A-scheme = quasicompact open subscheme of a projective A-scheme

$\boxed{4.5.11}$ $\mathbb{P}^0_A = \operatorname{Proj} A[T] \cong \operatorname{Spec} A$ as $\mathbb{P}^0_A$ is $D(T) = \operatorname{Spec} A$.

$\boxed{4.5.Q}$ $\mathbb{P}V := \operatorname{Proj}(\operatorname{Sym}^\bullet V^\vee)$ where V is k-vector space

$\operatorname{Sym}^\bullet V^\vee = k \oplus \operatorname{Sym} V^\vee \oplus \operatorname{Sym}^2 V^\vee \oplus \dots$

- Closed points of $\mathbb{P}V$ corresponds to maximal homogeneous prime of $\operatorname{Sym}^\bullet V^\vee$. Let such ideal be $P = P_0 \oplus P_1 \oplus \dots$

Every element in P_i must be generated by P_1 (i.e. if $x_1 \dots x_i \in P_i$ then one of x_j must be in P_1).

- P_1 is a subspace of $V^\vee$ so it's maximal when P_1 of dimension $\dim V - 1$. Every such space is $\{\vartheta \in V^\vee \mid \vartheta(a) = 0\}$ for some $a \in V$, $a \neq 0$.

$\Rightarrow$ P is generated by $\{\vartheta \in V^\vee \mid \vartheta(a) = 0\}$, which corresponds to 1-dim subspace $\{ka\}$ of V.

- Natural bijection between V and closed points of $\operatorname{Spec}(\operatorname{Sym}^\bullet V^\vee)$.

Closed points of $\operatorname{spec}(\operatorname{Sym}^\bullet V^\vee)$ corresponds to maximal ideals of $\operatorname{Sym}^\bullet V^\vee$, which are of the form

$\{\vartheta \in k \oplus V^\vee \mid \vartheta(a) = 0\}$ for some $a \in V$.

Chapter 5: Some properties of schemes

§5.1. Topological properties

5.1.A Suppose $\mathbb{P}^n_k = V(T) \cup V(U)$ for T, U homogeneous ideals of $S_+ = \{p \in k[x_0, \dots, x_n] \mid p(0, \dots, 0) = 0\}$

We have $D(x_i) = (V(T) \cap D(x_i)) \cup (V(U) \cap D(x_i))$

$$\operatorname{Spec} k[x_{j/i}]/(x_{i/i} - 1) = V(T') \cup V(U') \qquad (1)$$

where $T' = (T_{x_i})_0$ (T_{x_i} is homogeneous ideal in $k[x_0, \dots, x_n]_{x_i}$ obtained from T after localising at x_i).

$\Rightarrow$ T' generated as k-module by $\frac{a}{x_i^k}$ where $a \in T$ homogeneous of degree k.

From (1), we find $V(T'U') = \operatorname{Spec} \underbrace{k[x_{j/i}]/(x_{i/i} - 1)}_{\text{↳ integral domain}}$

$\Longrightarrow T'U' \subset (0) \Longrightarrow$ WLOG $T' = (0)$ as

so $T = 0$ (from description of T').

5.1.B Consider scheme X. We show there is bijection

$$\{\text{irreducible closed subsets of } X\} \longleftrightarrow \{\text{points in } X\}$$

\- Let p be point in X then its closure $\overline{\{p\}}$ is irreducible closed. Indeed, if $\overline{\{p\}} = U \cup V$ for closed V, U of $\overline{\{p\}}$ then WLOG $p \in V$, implying $\overline{\{p\}} \subset V \Rightarrow V = \overline{\{p\}}$.

\- Conversely, let T be irreducible closed of X.

Let $X = \bigcup U_i$ covered by affine open sets U_i, then $U_i \cap T$ open in T, hence irreducible. We have $T \cap U_i$ is irreducible closed in affine U_i so by 3.7E, this corresponds to a point $p_i \in U_i$. So that closure of $\{p_i\}$ in U_i is $T \cap U_i$, i.e. $T \cap U_i = U_i \cap \bigcap_{p_i \in V} V$ over closed V of X containing p_i.

We define the inverse map $T \mapsto p_i$.

\- To show the two maps are invertible.

We show closure of p_i in X is T. Indeed, suppose not then $T \setminus (T \cap U_i)$ and $\overline{\{p_i\}} = T \cap U_i$ are both closed, proper and cover T, which is irreducible, a contradiction.

\- To show $T \mapsto p_i$ is well-defined:

. If $U_i \cap U_j \neq 0$ then we show $p_i = p_j$. Indeed, let $U_{ij} = U_i \cap U_j$ then $T \cap U_{ij}$ open in $T \cap U_i$, hence must contain p_i. Else $T \cap (U_i \setminus U_{ij})$ is closed in $T \cap U_i$ and contains p_i, hence must contain closure of p_i in $T \cap U_i$, which is $T \cap U_i$ itself, a contradiction to $U_{ij} \neq \phi$. Thus $p_i \in U_j$. Because closure of p_i in X is T, its closure in U_j is $T \cap U_j$. But $T \cap U_j$ is also closure of p_j in U_j. As U_j affine, this implies $p_i = p_j$.

. Because T is irreducible, hence connected, which will give $p_i = p_j = p$ for all i, j.

5.1C It suffices to show if $X=\bigcup_{i=1}^{n} U_i$ is finite union of Noetherian subspaces then X is Noetherian.

Consider descending chain of closed subsets $I_1 \supset I_2 \supset \cdots$ then $(I_1 \cap U_i) \supset (I_2 \cap U_i) \supset \cdots$ descending chain of closed subsets of U_i, hence must stabilises.

Thus, after some large n, $I_n \cap U_i = I_{n+1} \cap U_i = \cdots$ for all $i \Longrightarrow I_n = I_{n+1} = \cdots$, as desired.

5.1D If X is quasicompact then as X has open cover of affine open sets, X is finite union of affine open set.

Conversely, if X is finite union of affine open set, i.e. $X = \bigcup_{i=1}^{n} V_i$ for affine V_i. Consider open cover $\{U_j\}$ of X. Then $\{U_j \cap V_i\}_{j \in I}$ is open cover of affine V_i, and because affine scheme is quasicompact. (3.6G), V_i is finite union of these.

Because of finiteness, we find X is finite union of U_i., as desired.

5.1E – We show if X quasicompact, then every point has a closed point in its closure.

Because X is quasicompact, $X = \bigcup_{i=1}^{n} U_i$ where U_i affine open of X. Note for affine scheme, because every prime ideal lies in a maximal ideal so every point in affine scheme has a closed point in its closure.

Consider point $P_1 \in X$. If P_1 not closed point then exists $P_2 \in \overline{\{P_1\}} \cap U_1$. Then closure of P_2 in U_1 has closed point $P_3 \in U_1$, i.e. $\overline{\{P_3\}} \cap U_1 = \{P_3\} \subset \overline{\{P_2\}} \cap U_1 \subset \overline{\{P_1\}} \cap U_1$.

If P_3 is not closed point then $\exists P_4 \in \overline{\{P_3\}} \cap U_2$ but $P_4 \notin U_1$. This follows $\overline{\{P_4\}} \cap U_1 = \phi$, or else $\overline{\{P_4\}} \cap U_1 \subset \overline{\{P_3\}} \cap U_1 = \{P_3\}$ and if equality holds then $\overline{\{P_3\}} = \overline{\{P_4\}} \Rightarrow P_3 = P_4$ by 5.1B, contradiction.

So far, we found $P_4 \in \overline{\{P_1\}}$ but $\overline{\{P_4\}} \cap U_1 = \emptyset$. Similarly as above, we find $P_5 \in \overline{\{P_4\}}$ but $\overline{\{P_5\}} \cap U_2 = \emptyset$. By keep going while assuming X has no closed points, because there are finitely many U_i, we arrive at contradiction that $\exists$ point P_n so $\{P_n\} \cap U_i = \emptyset \ \forall i$.

- Every nonempty closed subset of X contains a closed point of X. Consider closed V of X and point $x \in V$ then $\overline{\{x\}} \subset V$ and by previous result, x has closed point in its closure.

- Why 5.1E useful: If there is some property P of points that are "open" (i.e. point p has P then all points in nbh of p has P)., then to check if all points of quasicompact scheme has P, it suffices to check for closed points.

Indeed, if all closed points have P. Consider $x \in X$ then $\overline{\{x\}}$ contain closed point p. There exists nbh U of p st. all points in U have P. Note $U \cap \overline{\{x\}}$ nonempty open in $\overline{\{x\}}$, hence must contain x, else its complement, being closed, if contains x then also contains $\overline{\{x\}}$, which contradicts the fact $U \cap \overline{\{x\}}$ nonempty.

$\boxed{5.1F}$ - X quasiseparated scheme, i.e. intersection of any quasicompact open subsets is quasicompact.

- intersection of affine open, which are quasicompact must be quasicompact, hence finite union of affine open subsets.

- For the converse, because quasicompact open subset is finite union of affine open, and intersection of affine open is finite union of affine open, so intersection of quasicompact open sets is finite union of affine open. This is also a open subscheme so by 5.1D, this is quasicompact.

$\boxed{5.1G}$ Affine scheme are quasiseparated.

Suppose we have 2 affine open U, V of affine scheme X. Then U, V are quasicompact, hence finite union of $D(f)$, $f \in \mathcal{O}(X)$ $U = \bigcup_{i=1}^{n} D(f_i)$ and $V = \bigcup_{i=1}^{m} D(g_i)$. As $D(f_i) \cap D(g_j) = D(f_i g_j)$ so $U \cap V$ finite union of affine.

5.1H If X quasicompact, quasiseparated then by 5.1D and 5.1F, X is covered by finite number of affine open, any two of which are intersection also covered by finite number affine open.

Conversely, if $X = \bigcup_{i=1}^{n} U_i$ where U_i affine open and $U_i \cap U_j$ finite union of affine open. Then X is quasicompact. (proved before)

To show X is quasiseparated:

– We first show for affine U then $U \cap U_1$ is quasicompact

Because $U \cap U_i$ is open in U, hence union of affine W_{ij}

Because $U = \bigcup_i (U \cap U_i)$ so for each i, we can pick finitely many j so $U = \bigcup_{i,j} W_{ij}$ where $W_{ij} \subset U \cap U_i$.

We have $U \cap U_1 = \bigcup_{i,j} W_{ij} \cap U_1 = \bigcup_{j,i} (W_{ij} \cap U_1 \cap U_i)$

Note $U_1 \cap U_i$, W_{ij} are quasicompact in U_i so their intersection also quasicompact $\Rightarrow$ $U \cap U_1$ quasicompact.

– for affine U, V then $U \cap V \cap U_i = (U \cap U_i) \cap (V \cap U_i)$ is quasicompact by quasicompactness of U_i.

$\Rightarrow$ $U \cap V$ is quasicompact $\Rightarrow$ X quasiseparated.

5.1I Projective A-scheme is quasicompact, quasiseparated

$S_\bullet$ is finitely generated graded ring over A. Let $f_1, \dots, f_n$ generates $S_\bullet$ of pos deg. homogeneous pols. Then by 4.5I, $\bigcup D(f_i) = \operatorname{Proj} S_\bullet$ and note $D(f_i) \cap D(f_j) = D(f_i f_j)$.

5.1J $X = \operatorname{Spec} k[x_1, x_2, \dots]$. Let $U = X \setminus [m]$ where $m = (x_1, x_2, \dots)$. Take two copies of X and glue along U. The resulting scheme is not quasiseparated as U is not quasicompact.

§5.2 Reducedness and integrality.

5.2A – Scheme reduced iff none of the stalks have nonzero nilpotents. Direction "⇒" is clear by def of stalks. For the converse, note $\mathcal{O}_X(U) \hookrightarrow \prod_{p \in U} \mathcal{O}_{X,p}$ ring hom so $\mathcal{O}_X(U)$ is also reduced.

– If X reduced, $f, g \in \mathcal{O}_X(X)$ so that f, g agree at all points then $f = g$. Indeed, it suffices to show this for $X = \operatorname{Spec} A$ (to prove for general X, cover it by affine open $D(f)$ then as $f|_{D(f)} = g|_{D(f)}$, we find $f = g$ by def of sheaf).

Thus, we have $f = g \bmod p$ for all $p \in \operatorname{Spec} A$ so $f - g$ lies in intersection of all primes of A, which is nilradical of A. As X reduced, $\mathcal{O}_X(\operatorname{Spec} A) = A$ is reduced, so A has no nilpotent $\Rightarrow f = g$ in A.

5.2B If A reduced ring then show $\operatorname{Spec} A$ is reduced.

Because stalk at prime p, i.e. A_p is reduced so $\operatorname{Spec} A$ is reduced. $\Rightarrow \mathbb{A}^n_k$ is reduced as $k[x_1, \dots, x_n]$ reduced. $\mathbb{P}^n_k$ is reduced as it is covered by reduced affine scheme (which have reduced stalks).

5.2C $(k[x,y]/(y^2, xy))_x$ has no nonzero nilpotent element.

We visualise $k[x,y]/(y^2, xy)$ as x-axis with 1st order circle at origin. When localising at x means removing parts where x vanishes $\Rightarrow$ just x-axis without origin $\Rightarrow \operatorname{Spec} k[x]_x$.

Indeed, consider map $(k[x,y]/(y^2, xy))_x \longrightarrow k[x]_x$

$$\begin{aligned} y &\longmapsto 0 \\ f(x) &\longmapsto f(x). \end{aligned}$$

Note that LHS has $y = 0$ as $\frac{0}{x} = y$.

5.2D